Practice Midterm Exam Math 220

Problem 1. Use Mathematics Induction to show: $\forall n \geq 4, n \in \mathbb{N}, n! \geq n^2$. (Recall: $n! = 1 \cdot 2 \cdot \ldots \cdot n$)

Proof ① When $n = 4$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$4^2 = 16$

$\therefore \quad 4! > 4^2$

② Assume $k! \geq k^2$. Then

$(k+1)! = (k+1) \cdot k! \geqslant (k+1) \cdot k^2$

Because $k(k-1) \geqslant 1$ for $k \geqslant 4$

$\therefore \quad k^2 \geqslant k+1$

$\therefore \quad (k+1)! \geqslant (k+1) \cdot (k+1) = (k+1)^2$

So by Induction principle, we have

$n! \geqslant n^2$ for all $n \geq 4$.

Problem 2. For each set S below, determine:
(1) $\text{int}S$, (2) $\text{bd}S$, (3) $\max S$, (4) $\sup S$, (5) the set S' of the accumulation points of S.

(a) $S = [-1, 2) \cup (2, 3)$

$\text{int } S = (-1, 2) \cup (2, 3)$
$\text{bd } S = \{-1, 2, 3\}$
$\max S$: no
$\sup S$: 3
S' : $[-1, 3]$

(b) $S = \bigcup_{n=2}^{\infty} \left[-\frac{1}{n}, \frac{2}{n}\right)$ Note $S = [-\frac{1}{2}, 1)$

$\text{int } S = (-\frac{1}{2}, 1)$
$\text{bd } S = \{-\frac{1}{2}, 1\}$
$\max S$: no
$\sup S$: 1
S' : $[-\frac{1}{2}, 1]$

(c) $S = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ Note: $S = \{\cdots, 1-\frac{1}{5}, 1-\frac{1}{3}, 0, 1+\frac{1}{2}, 1+\frac{1}{4}, \cdots\}$

$\text{int } S$: no
$\text{bd } S$: $S \cup \{1\}$
$\max S$: $1+\frac{1}{2}$
$\sup S$: $1+\frac{1}{2}$
S' : 1

(d) $S = \{p \in \mathbb{R} \backslash \mathbb{Q} : 2 < p < 2\sqrt{2}\}$

$\text{int } S$: ϕ
$\text{bd } S$: $[2, 2\sqrt{2}]$
$\max S$: no
$\sup S$: $2\sqrt{2}$
S' : $[2, 2\sqrt{2}]$

Problem 3. Mark "True" or "False" to each of the following statement. If true, provide a proof; if false, provide a counterexample.

(a) Let S, T be subsets of $\mathbb{R}$. If $S \subseteq T$, then the set of accumulation points of S is contained in the set of accumulation points of T.

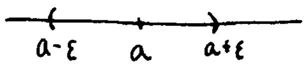
$\boxed{\text{True}}$ If x is an accu. pt of S, then $\forall \varepsilon > 0$, $N^*(x;\varepsilon) \cap S \neq \phi$

Since $S \subseteq T$, $N^*(x;\varepsilon) \cap T \neq \phi$

$\therefore$ x is also an accum. pt. of T.

(b) Let S be a subset of $\mathbb{R}$ and let a be a lower bound of S. If $a \in S$, then S is not open.

$\boxed{\text{True}}$ If S is open, since $a \in S$, $\exists$ nbhd $N(a;\varepsilon)$ s.t. $N(a;\varepsilon) \subseteq S$. In particular, $a - \frac{\varepsilon}{2} \in S$



This contradicts to a is a lower bound of S.

$\therefore$ S is not open.

(c) Let S, T be subsets in $[0,1)$ and $\sup T \in [0,1)$. Then $\sup \left\{ \frac{s}{1-t} : s \in S, t \in T \right\}$ exists in $\mathbb{R}$.

$\boxed{\text{True}}$ $\because$ $\sup T \in [0,1)$,

$\therefore$ $t \leq \sup T < 1$. for any $t \in T$.

$\therefore$ $\frac{s}{1-t} \leq \frac{1}{1-\sup T}$ for any $s \in S$, $t \in T$.

$\therefore$ The set $\left\{ \frac{s}{1-t} : s \in S, t \in T \right\}$ is bounded above and the Completeness Axiom of $\mathbb{R}$ implies $\sup \left\{ \frac{s}{1-s} : s \in S, t \in T \right\}$ exists in $\mathbb{R}$.

Problem 4. Let $S_1, S_2, ..., S_n$ be subsets of $\mathbb{R}$. Prove that

$$\text{int}\,(S_1 \cap S_2 \cap ... \cap S_n) = (\text{int}\,S_1) \cap (\text{int}\,S_2) \cap ... \cap (\text{int}\,S_n).$$

Proof. "$\subseteq$": $\forall\ x \in \text{int}\,(S_1 \cap S_2 \cap \cdots \cap S_n)$.

$\exists$ nbhd $N(x;\varepsilon)$, such that

$N(x;\varepsilon) \subseteq S_1 \cap \cdots \cap S_n$

$\therefore$ $N(x;\varepsilon) \subseteq S_1 \Rightarrow x \in \text{int}\,S_1$

$N(x;\varepsilon) \subseteq S_2 \Rightarrow x \in \text{int}\,S_2$

$\vdots$

$N(x;\varepsilon) \subseteq S_n \Rightarrow x \in \text{int}\,S_n$.

$\therefore$ $x \in (\text{int}\,S_1) \cap \cdots \cap (\text{int}\,S_n)$

"$\supseteq$": $\forall\ x \in (\text{int}\,S_1) \cap \cdots \cap (\text{int}\,S_n)$.

Since $x \in \text{int}\,S_1$, $\exists\ N(x;\varepsilon_1)$, s.t.

$N(x;\varepsilon_1) \subseteq S_1$

$x \in \text{int}\,S_2$, $\exists\ N(x;\varepsilon_2)$, s.t.

$N(x;\varepsilon_2) \subseteq S_2$

$\vdots$

$x \in \text{int}\,S_n$, $\exists\ N(x;\varepsilon_n)$, s.t.

$N(x;\varepsilon_n) \subseteq S_n$

Let $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_n\}$. $\therefore$ $\varepsilon > 0$ and

$N(x;\varepsilon) \subseteq N(x;\varepsilon_i) \subseteq S_i$. for $i = 1, \cdots, n$.

$\therefore$ $N(x;\varepsilon) \subseteq S_1 \cap \cdots \cap S_n$

$\therefore$ $x \in \text{int}\,(S_1 \cap \cdots \cap S_n)$