Homework Set 2, Math 526
Due: February 9

(1) P.83-85: #7, #8, #9, #11

(2) Let (M, g) be a Riemannian manifold. A vector field X on M is a *Killing* vector field if

$$(L_X g)(Y, Z) := g((\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

for all Y, Z.

(a) If X is a Killing vector field, show that ∇X is skew-symmetric as a linear map from T_pM to T_pM for any $p \in M$.

(b) Show: X is a Killing vector field iff the local 1-parameter group associated to X consists of local isometries of (M, g).

Remark. Any vector field Y induces an ODE $\phi' = Y(\phi)$. The ODE has a unique solution $\phi(t, x)$ for each $x \in M$ with $\phi(0, x) = x$ on $[0, T(x))$. For $x_0 \in M$ there exists a neighborhood U of x_0 s.t. $\phi(t, x)$ exists on $[0, T_0)$ for all $x \in U$ and $(t, x) \to \phi(t, x)$ is smooth on $[0, T_0) \times U$. For each t, $\phi(t, \cdot)$ is a local diffeomorphism, and $\phi(t_1, \phi(t_2, x)) = \phi(t_1 + t_2, x)$. The local diffeomorphisms $\phi(t, \cdot)$ form a local 1-parameter group.

(c) The Lie bracket of two Killing vector fields is still a Killing vector field.

(d) Show that the vector fields $x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$ are Killing vector fields on $(\mathbb{S}^n, g_{can})$.