Homework Set 1, Math 526
Due: January 24, 2016

(1) Let ∇ be an affine connection on an n-dimensional manifold M. Let $T : TM \times \cdots \times TM \to C^\infty(M)$ be a $(r,0)$-type tensor. The covariant derivative ∇T of T is a tensor of $(r+1,0)$-type defined by

$$\nabla T(X_1, ..., X_r, X) = X(T(X_1, ..., X_r)) - T(\nabla_X X_1, X_2, ..., X_r) - \cdots - T(X_1, ..., \nabla_X X_r).$$

For $X \in \Gamma(TM)$, the covariant derivative $\nabla_X T$ of T relative to X is a $(r,0)$-type tensor given by

$$\nabla_X T(X_1, ..., X_r) = \nabla T(X_1, ..., X_r, X).$$

Now let ∇ be the Levi-Civita connection of (M, g). Show $\nabla g = 0$.

(2) page 56: 1,3,7,8.

(3) Consider the two parametrizations of T^2:
  (a) $\phi_1(\alpha, \beta) = (e^{\sqrt{-1}\alpha}, e^{\sqrt{-1}\beta}) \subset \mathbb{R}^4$
  (b) $\phi_2(\alpha, \beta) = ((2 + \cos\alpha)\cos\beta, (2 + \cos\alpha)\sin\beta, \sin\alpha) \subset \mathbb{R}^3$.

Hence T^2 is equipped with two Riemannian submanifold structures. In each case, compute $[\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}], \nabla_{\frac{\partial}{\partial\alpha}}\frac{\partial}{\partial\beta}$. (You may use the previous problem to avoid the computation of the Christoffel symbols)

(4) The Klein bottle K: take a square $\{0 \leq x, y \leq 1\}$ in $\mathbb{R}^2$ and identify the opposite sides by $(0, y) \sim (1, y), (x, 0) \sim (1 - x, 1)$, the result is a closed surface.
  (a) Prove the Klein bottle is nonorientable. (hint: consider the change of the orientation in the tangent planes along the loop $c(y) = (1/2, y), y \in [0, 1]$.
  (b) Let M be the submanifold in the Klein bottle defined by $1/4 \leq x \leq 3/4, 0 \leq y \leq 1$. Is M orientable? Can M be embedded in $\mathbb{R}^3$?