A NOTE ON CLOSED STABLE GEODESICS IN HYPERKÄHLER FOUR DIMENSIONAL MANIFOLDS

1. introduction

Geodesics are the critical points of the length functional. A geodesic is stable if its second variation is nonnegative. A stable geodesic is a local minima of the length functional among curves in its homotopy class. The second variation of length involves the sectional curvatures of the ambient space M. When the sectional curvatures are nonpositive, all closed geodesics in M are stable, however, this is no longer true if M has positive sectional curvatures. It is hard, in general, to get information from the second variation if K changes sign on different sections.

In this short note, we make a simple observation that existence of a closed stable geodesic in a four dimensional hyperkähler manifold M imposes geometric restriction on M, namely, the Riemann curvature tensor R of M must be zero along the geodesic. This rules out closed stable geodesics if R is nowhere vanishing, e.g. the Eguchi-Hanson space.

Our investigation arises from reading the interesting paper [2]. The authors of [2] have conjectured that in a compact Calabi-Yau manifold N there exists a closed stable stable geodesic and moreover the number of the isolated closed stable closed geodesics of length less than L grows asymptotically as $L^{\dim N}$, and they have provided a physical argument for non-existence of nontrivial closed stable geodesics in the Eguchi-Hanson space.

2. Result and proof

Our result is as follows.

Theorem. *Let M be a hyperkähler manifold of dimension 4. If there exists a nontrivial closed stable geodesic γ in M, then the Riemann curvature tensor of M vanishes along γ. In particular, there are no nontrivial closed stable goedesics in the Eguchi-Hanson space (in*

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fact, in any hyperkähler 4-manifold whose Riemann curvature tensor is nowhere vanishing).

Proof. Let R be the Riemannian curvature tensor of M and let $I, J, K = IJ$ be the parallel complex structures with respect to the covariant derivative ∇ on M which determine the hyperähler structure compatible with the Riemannian metric on M. Parametrize γ by its arc-length and consider the unit vector fields $I\gamma', J\gamma', K\gamma'$ along γ.

From the second variation formula of length at a closed geodesic and the assumption that γ is stable, we have the stability inequality for any variation vector field X of γ:

$$0 \leq \int_\gamma \left(|\nabla_{\gamma'} X|^2 - \langle R(\gamma', X)\gamma', X\rangle\right) := \mathrm{I}_\gamma(X, X).$$

Substituting X with $I\gamma', J\gamma', K\gamma'$ into the stability inequality above and taking summation:

$$\begin{aligned} 0 &\leq \mathrm{I}_\gamma(I\gamma', I\gamma') + \mathrm{I}_\gamma(J\gamma', J\gamma') + \mathrm{I}_\gamma(K\gamma', K\gamma') \\ &= \int_\gamma \left(|\nabla_{\gamma'}(I\gamma')|^2 + |\nabla_{\gamma'}(J\gamma')|^2 + |\nabla_{\gamma'}(K\gamma')|^2\right) \\ &\quad - \left(\langle R(\gamma', I\gamma')\gamma', I\gamma'\rangle + \langle R(\gamma', J\gamma')\gamma', J\gamma'\rangle + \langle R(\gamma', K\gamma')\gamma', K\gamma'\rangle\right) \\ &= -\int_\gamma Ric\,(\gamma', \gamma') \\ &= 0 \end{aligned}$$

where we have used

$$\begin{aligned} \nabla_{\gamma'}(I\gamma') &= I\,\nabla_{\gamma'}\gamma' = 0, \\ \nabla_{\gamma'}(J\gamma') &= J\,\nabla_{\gamma'}\gamma' = 0, \\ \nabla_{\gamma'}(K\gamma') &= K\,\nabla_{\gamma'}\gamma' = 0 \end{aligned}$$

since I, J, K are parallel with respect to ∇ and γ is a geodesic and that $I\gamma', J\gamma', K\gamma', \gamma'$ form an orthonormal frame along γ and M is Ricci flat. Therefore, since each of the three terms on the right hand side of the inequality above is nonnegative due to stability of γ, we have

$$\mathrm{I}_\gamma(I\gamma', I\gamma') = \mathrm{I}_\gamma(J\gamma', J\gamma') = \mathrm{I}_\gamma(K\gamma', K\gamma') = 0.$$

Stability of γ then implies that the first eigenvalue λ_0 of the Jacobi operator is nonnegative:

$$\lambda_0 = \inf_X \frac{I_\gamma(X, X)}{\int_\gamma |X|^2} \geq 0$$

for any variation field X along γ which is not identically zero. Therefore $\lambda_0 = 0$ and $I\gamma', J\gamma', K\gamma'$ are eigenfunctions for λ_0, hence they are Jacobi vector fields along γ. Since the vector fields $I\gamma', J\gamma', K\gamma'$ are parallel along γ, the first term in the Jacobi equation

$$\frac{D^2X}{dt^2} + R(\gamma'(t), X(t))\gamma'(t) = 0$$

vanishes for $X = I\gamma', J\gamma', K\gamma'$, and this leads to

$$R(\gamma', I\gamma')\gamma' = R(\gamma', J\gamma')\gamma' = R(\gamma', K\gamma')\gamma' = 0.$$

By the Kähler identities for curvature [4]:

$$R(JX, JY) = J \circ R(X, Y) \quad \text{and} \quad R(X, Y) \circ J = J \circ R(X, Y)$$

we see the sectional curvature

$$\begin{aligned} \langle R(J\gamma', I\gamma')J\gamma', I\gamma'\rangle &= \langle R(J\gamma', JK\gamma')J\gamma', I\gamma'\rangle \\ &= -\langle R(\gamma', K\gamma')\gamma', I\gamma'\rangle \\ &= 0. \end{aligned}$$

Similarly,

$$\langle R(K\gamma', I\gamma')K\gamma', I\gamma'\rangle = 0.$$

We conclude that the sectional curvatures vanish on the sections containing $I\gamma'$. The same reasoning shows all sectional curvatures vanish on sections containing $J\gamma'$ and on those containing $K\gamma'$. It then follows that the Riemann curvature tensor R vanishes along γ [1].

Since the Eguchi-Hanson space is a hyperkähler 4-dimensional manifold whose Riemann curvature tensor is nowhere vanishing [3], it does not admits any nontrivial closed stable geodesics. □

Remark. *After this note was written, I was informed that the result in this note was obtained by J.-P. Bourguignon and S.-T. Yau in "Sur les métriques riemanniennes à courbure de Ricci nulle sur le quotient d'une surface K3", C. R. Acad. Sci. Paris Ser A-B 277 (1973), A1175-A1177.*

References

[1] J. Cheeger, D. Ebin, Comparison Theorems in Riemannian Geometry, AMS Chelsea Publishing, 2008.

[2] M. Douglas, P. Gao, Geodesics on Calabi-Yau manifolds and winding states in nonlinear sigma models, arXiv:1301.1687v1.

[3] T. Eguchi, A.J. Hanson, Asymptotically flat solutions to Euclidean gravity, Physics Letters, 74B (1978) 249-251.

[4] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, II, John Willey & Sons, Inc, 1969.