Homework Set 3, Math 526
Due: March 17 (Tuesday), 2015

(1) Let M be a Riemannian manifold s.t. for any two points $p, q \in M$ the parallel transport from p to q does not depend on the curve connecting p, q. Prove the curvature of M is zero. (Hint: use a parametrized surface $f : U \subset \mathbb{R}^2 \to M$ and a vector field along f. We showed $\frac{D}{\partial s}\frac{D}{\partial t}V - \frac{D}{\partial t}\frac{D}{\partial s}V = R(f_s, f_t)V$)

(2) Let $c : [0, l] \to M$ be a geodesic and X be a vector field on M s.t. $X(c(0)) = 0$. Show that

$$\nabla_{c'}(R(c', X)c')(0) = (R(c', X')c')(0),$$

where $X' = DX/dt$. (Hint: compute $(\nabla_{c'}R)(c', X, c', Z)$ at $t = 0$)

(3) Let M be a locally symmetric space (i.e. $\nabla R = 0$). Let $c : [0, l] \to M$ be a geodesic.
   (a) If X, Y, Z are parallel vector fields along c, prove that $R(X, Y)Z$ is parallel along c.
   (b) Perove that if M has constant sectional curvature, then M is locally symmetric.

(4) Let M be a Riemannian manifold with nonpositive sectional curvature. Prove that the conjugate locus $C(p)$ is empty, for any $p \in M$.

(5) Let c be a geodesic in a locally symmetric space M, $V = c'(0), p = c(0)$. DefineK_v : $T_pM \to T_pM$ by

$$K_V(X) = R(V, X)V.$$

   (a) Prove that K_V is self-adjoint.
   (b) Choose an orthonormal basis $e_1, ..., e_n$ of T_pM which diagonalize K_V:

$$K_V(e_i) = \lambda_i e_i, \quad i = 1, ..., n.$$

   Parallelly translating $e_1, ..., e_n$ along c, show

$$K_{c'(t)}(e_i(t)) = \lambda_i e_i(t).$$

   (c) Let $J(t) = \sum X^i(t)e_i(t)$ be a Jacobi field along c. Show the Jacobi equation is equivalent to

$$X^i_{tt} + \lambda_i X^i = 0, \ \ i = 1, ..., n.$$

   (d) Show the conjugate points of p along c are given by $c(k\pi/\sqrt{\lambda_i})$, where k is a positive integer and λ_i is a positive eigenvalue of K_V.