Homework Set 2, Math 526
Due: February 24 (Tuesday), 2014

(1) Let $= \{(x, y) \in \mathbb{R}^2 : y > 0\}$ be the upper half plane equipped with the metric $g = \frac{dx^2+dy^2}{y^2}$. Set $z = x + iy$ and $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{R}, ad - bc = 1$. Show that f is an isometry of M.

(2) Show that a geodesic of $(\mathbb{R}P^n, g_{can})$, where g_{can} is the metric given by the canonical metric on $\mathbb{S}^n$ via the 2:1 Riemainnian covering, is minimal if and only if its length is less than or equal to $\pi/2$.

(3) Let f, g be two isometries of a connected Riemannian manifold (M, g). If $f(p) = g(p), df_p = dg_p$, show that $f = g$.

(4) Let M be a Riemannian manifold of dimension n. For any $p \in M$, show that there exists a neighbourhood U of p in M and n vector fields $E_1, ..., E_n$ over U such that they are orthonormal at each point in U and $\nabla_{E_i} E_j(p) = 0$ (Such a family is called a geodesic frame at p). Is it true, in general, that $\nabla_{E_i} E_j = 0$ everywhere in U?

(5) Let (M, g) be a Riemannian manifold. A vector field X on M is a *Killing* vector field if

$$(L_X g)(Y, Z) := g((\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

for all Y, Z.

(a) If X is a Killing vector field, show that ∇X is skew-symmetric as a linear map from the tangent space at a point to itself.

(b) Show: X is a Killing vector field iff the local 1-parameter group associated to X (see, page 22, 1.C.3 in textbook) consists of local isometries of (M, g).

(c) The Lie bracket of two Killing vector fields is still a Killing vector field.

(d) Show that the vector fields $x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$ are Killing vector fields on $(\mathbb{S}^n, g_{can})$.