Homework Set 1, Math 526
Due: January 29, 2014

(1) Let ∇ be an affine connection on an n-dimensional manifold M. Let $T : TM \times \cdots \times TM \to C^\infty(M)$ be a $(r, 0)$-type tensor. The covariant derivative ∇T of T is a tensor of $(r+1, 0)$-type defined by

$$\nabla T(X_1, ..., X_r, X) = X(T(X_1, ..., X_r)) - T(\nabla_X X_1, X_2, ..., X_r) - \cdots - T(X_1, ..., \nabla_X X_r).$$

For $X \in \Gamma(TM)$, the covariant derivative $\nabla_X T$ of T relative to X is a $(r, 0)$-type tensor given by

$$\nabla_X T(X_1, ..., X_r) = \nabla T(X_1, ..., X_r, X).$$

Now let ∇ be the Levi-Civita connection of (M, g). Show $\nabla g = 0$.

(2) Write a detailed proof of Proposition 2.56 in the textbook (the proof there is almost complete).

(3) 2.57 and write down the Riemannian metrics in the given coordinates for both a) and b).

(4) The Klein bottle K: take a square $\{0 \le x, y \le 1\}$ in $\mathbb{R}^2$ and identify the opposite sides by $(0, y) \sim (1, y), (x, 0) \sim (1 - x, 1)$, the result is a closed surface.

  (a) Prove the Klein bottle is nonorientable. (hint: consider the change of the orientation in the tangent planes along the loop $c(y) = (1/2, y), y \in [0, 1]$.

  (b) Let M be the submanifold in the Klein bottle defined by $1/4 \le x \le 3/4, 0 \le y \le 1$. Is M orientable? Can M be embedded in $\mathbb{R}^3$?

  (c) Read (you do not need to write it down) the discussion in the textbook (page 82-83) on geodesics of K with the flat metric.

(5) Let $(\widetilde{M}, \widetilde{g}), (M, g)$ be two Riemannian manifolds of the same dimension. A map $p : \widetilde{M} \to M$ is a Riemannian covering map if (i) p is a smooth covering map and (ii) p is a local isometry. For a Riemannian covering map p, $\widetilde{g}$ is necessarily the pullback metric of g by p : $\widetilde{g}(X, Y) = g(p_* X, p_* Y), \forall X, Y \in T_x\widetilde{M}, \forall x \in \widetilde{M}$ (see p.59).

Let a_1, a_2 be a basis of $\mathbb{R}^2$. The basis defines a lattice $\Gamma = \{k_1 a_1 + k_2 a_2 : k_1, k_2 \in \mathbb{Z}\}$. Define $T_\Gamma = \mathbb{R}^2/\Gamma$ (elements in Γ are viewed as translations) and $p : \mathbb{R}^2 \to \mathbb{T}^2$ by

$$p(x_1 a_1 + x_2 a_2) = (e^{2\sqrt{-1}\pi x_1}, e^{2\sqrt{-1}\pi x_2}).$$

  (a) Show p induces a diffeomorphism $\hat{p}$ from $\mathbb{R}^2/\Gamma$ to $\mathbb{T}^2$.

  (b) Let $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ be the smooth covering map given by the projection. Let $\hat{g}_\Gamma$ be the metric on $\mathbb{R}^2/\Gamma$ such that π is a Riemannian covering map when $\mathbb{R}^2$ is equipped with $\langle , \rangle$. Then $\hat{p}$ is isometric from $(\mathbb{R}^2/\Gamma, \hat{g}_\Gamma) \to (\mathbb{T}^2, g_\Gamma)$ where $g_\Gamma = (\hat{p}^{-1})^* \hat{g}_\Gamma$. Check that $p^* g_\Gamma = \langle , \rangle$.

  (c) If there exists a diffeomorphism $f : (\mathbb{T}^2, g_{\Gamma_1}) \to (\mathbb{T}^2, g_{\Gamma_2})$ that satisfies

$$f^* g_{\Gamma_2} = \varphi g_{\Gamma_1}$$

  where φ is a positive function on T_{Γ_1} (such a f is called a *conformal* diffeomorphism), what can you conclude about Γ_1, Γ_2?