• (a) Show, by direct computation, that for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $f(x, y) = x^2$ that

$$\frac{\partial f}{\partial x} \frac{\partial}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial}{\partial y} \neq \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}$$

(1)

but

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta.$$

(b) The euclidean inner product is

in $(x, y)$ coordinate: $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

in $(r, \theta)$ coordinate: $g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$. Let $g^{ij}$ be the $(i, j)$-entry of the inverse matrix $g^{-1}$. Show, by direct computation, that

$$g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

is independent of the two coordinate systems.

• page 201: 8-10, 8-16

• Use local coordinate charts, prove

  (a) the Jacobi identity for the Lie bracket.
  (b) For a diffeomorphism $f : M \rightarrow N$, $f_*[X, Y] = [f_*X, f_*Y]$ for smooth vector fields $X, Y$ on $M$.

• page 199: 8-1 (review the extension result for smooth functions, page 45)