Homework Set 4, Math 424
Due: March 12, 2015

(1) Let $\phi : U \subset \mathbb{R}^2 \to S$ be a parametrization of a surface S. $E = \langle \phi_u, \phi_u \rangle, F = \langle \phi_u, \phi_v \rangle, G = \langle \phi_v, \phi_v \rangle$. Let V be a vector in $T_p\mathbb{R}^3$ where $p \in \phi(U)$. Write the projection V^T of V to the tangent plane T_pS in terms of E, F, G and ϕ_u, ϕ_v (but not the cross product).

(2) Read the solution in the textbook for Exercise (12), page 96 and page 89. This shows that K can be written in terms of E, F, G and their derivatives, at least in orthogonal parametrization (i.e. $F \equiv 0$), and this is true in general parameterization as well (but more complicated) hence proves Gauss's theorema egregium. Show that there does not exist a surface with $E = G = 1, F = 0$ and $e = 1, g = -1, f = 0$ for any parametrization.

(3) Given two surfaces

$$\begin{aligned} \phi(u,v) &= (u\cos v, u\sin v, \log u) \\ \psi(u,v) &= (u\cos v, u\sin v, v) \end{aligned}$$

show that $K(\phi(u,v)) = K(\psi(u,v))$ but $\psi \circ \phi^{-1}$ is not an isometry.

(4) Let S be a surface of revolution defined by $F(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$.
  (a) Compute E, F, G, e, f, g for S.
  (b) Write down formulas for the Gauss curvature K and the mean curvature H.
  (c) For the curve C given by $(0, f(v), g(v))$ in the yz-plane, if v is the arc length parameter, simplify your formulas for K and H.
  (d) Using (c), Study surfaces of revolution with $K \equiv 1$, or 0, or -1.

(5) Show that if a compact orientable surface S whose Gauss map $N : S \to \mathbb{S}^2$ is an isometry then S must be $\mathbb{S}^2$. (You can use the fact that there exists no compact surface in $\mathbb{R}^3$ such that $H \equiv 0$. Note that S is not assumed to be a surface of revolution)