Homework Set 2, Math 424
Due: February 5, 2014

(1) Let S be the union of the two cones $C_1 = \{(x, y, 1 - \sqrt{x^2 + y^2}) : 0 < x^2 + y^2 \leq 1\}$ and $C_2 = \{(x, y, -1 + \sqrt{x^2 + y^2}) : 0 < x^2 + y^2 \leq 1\}$ with the two vertices deleted. Show that S is not a regular surface in $\mathbb{R}^3$. (hint: read the solution to Exercise 2.24 in the textbook)

(2) Let $U = \{(u, v) : 0 < u < 2\pi, 0 < v < 2\pi\}$ and $f : U \to \mathbb{R}^3$ be the function

$$f(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u)$$

where $a > r > 0$.

(a) Show that $(U; f)$ is a parametrization of the torus given by rotating, about the z-axis, of the circle $(y - a)^2 + z^2 = r^2$.

(b) At the point $p = f(\pi/3, \pi/6)$, find the equation that defines the tangent plane $T_p f(U)$.

(c) Compute the first fundamental form w.r.t. this parametrization (i.e. find E, F, G).

(3) Show that the tangent planes of a surface given by the graph of $z = xf(y/x), x \neq 0$, where f is a differentiable function, all pass through the origin $(0, 0, 0)$.

(4) Let $F : S_1 \to S_2, G : S_2 \to S_3$ be diffeomorphisms between regular surfaces. Prove that $G \circ F : S_1 \to S_3$ is a diffeomorphism (read the last paragraph on page 43 of the textbook).

(5) Let $\alpha : (a, b) \to S$ be a regular curve in a regular surface S in $\mathbb{R}^3$. Let $(U; \phi)$ be a parametrization of S around a point $\alpha(s_0) \in S$, $s_0 \in (a, b)$. Let $\beta(s) = \phi^{-1} \circ \alpha(s)$, $\beta(s) = (\beta_1(s), \beta_2(s)) \in U$. Prove

$$|\alpha'(s)|^2 = E\beta_1'(s)^2 + 2F\beta_1'(s)\beta_2'(s) + G\beta_2'(s)^2$$

where E, F, G are the coefficients of the first fundamental form w.r.t. the chart $(U; \phi)$.

(6) Show the set $\{(x, y, z) : \cos x + \cos y + \cos z = 0\}$ is a regular surface.