## Homework Set 1, Math 424 <br> Due: January 22, 2014

(1) Let $\phi: J \rightarrow I$ be a diffeomorphism between intervals $I$ and $J$ (i.e. $\phi$ is differentiable, one-to-one and onto). Let $\alpha: I \rightarrow \mathbb{R}^{3}$ be a space curve. Given $[a, b] \subseteq J$ with $\phi([a, b])=[c, d]$, prove that the length of $\alpha \circ \phi$ over $[a, b]=$ length of $\alpha$ over $[c, d]$.
(2) For the logarithmic spiral $\alpha: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ defined by

$$
\alpha(t)=\left(a e^{b t} \cos t, a e^{b t} \sin t\right)
$$

with $a>0, b<0$, compute the arc-length function $s(t)$ from $t_{0}$. Reparametrize $\alpha$ by its arc-legnth and study its trace.
(3) Let $\alpha: I \rightarrow \mathbb{R}^{2}$ be a curve parametrized by arc-length (p.b.a.l.). If there is a differentiable function $\theta: I \rightarrow \mathbb{R}$ that is the angle of the tangent line to $\alpha$ makes with a fixed direction, show that $\theta^{\prime}(s)= \pm k(s)$, where $k$ is the curvature of $\alpha$.
(4) Show that a curve $\alpha: I \rightarrow \mathbb{R}^{3}$ p.b.a.l. with $k(s)>0$ for all $s \in I$ is planar iff its torsion $\tau \equiv 0$.
(5) Let $\alpha: I \rightarrow \mathbb{R}^{2}$ be a curve p.b.a.l. and $s_{0} \in I$. Define $f: I \rightarrow \mathbb{R}$ by

$$
f(s)=\left\langle\alpha(s)-\alpha\left(s_{0}\right), N(s)\right\rangle
$$

where $N$ is the unit normal vector such that $\left\{\alpha^{\prime}, N\right\}$ is positively oriented. Prove: $f\left(s_{0}\right)=0, f^{\prime}\left(s_{0}\right)=0, f^{\prime \prime}\left(s_{0}\right)=k\left(s_{0}\right)$ and then prove
(a) If $k\left(s_{0}\right)>0$ then there is a neighborhood $J$ of $s_{0}$ in $I$ such that $\alpha(J)$ belongs to the closed half-plane with the tangent line of $\alpha$ at $s_{0}$ as its border and $N\left(s_{0}\right)$ points inside.
(b) If there is a neighborhood $J$ of $s_{0}$ in $I$ such that $\alpha(J)$ is contained in the closed half-plane described above, then $k\left(s_{0}\right) \geq 0$.
(6) Suppose that a curve $\alpha: I \rightarrow \mathbb{R}^{3}$ is p.b.a.l. and has positive curvature. Show that all of the tangent lines of $\alpha$ make a constant angle with a given direction iff $\tau(s)=a k(s)$ for some constant $a \in \mathbb{R}$.

