Homework 7 Solutions

Prove or disprove the following statements:

1. Let $A$, $B$ and $C$ be sets. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   **Proof:** Since this is a set equality, we need to prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

   **Proof of $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$:** Assume that $x \in A \cap (B \cup C)$. Then we know that $x \in A$ and $x \in (B \cup C)$. That means $x \in A$ and moreover, $x \in B$ or $x \in C$. Then we have two cases, $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$. First assume $x \in A$ and $x \in B$. Thus, $x \in (A \cap B)$. This implies $x \in (A \cap B) \cup (A \cup C)$. We see that the case $x \in A$ and $x \in C$ is treated similarly to the previous case. Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

   **Proof of $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$**:

   $A \cap (B \cup C) = (A \cap B \cup A \cap C) \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

   Therefore, we see that for $A$, $B$ and $C$ be sets. Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2. If $A$ and $B$ are sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

   **Proof:** This is a set equality, so we have to prove $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

   **Proof of $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$**:

   Let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then we see that $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$. This implies that $X \subseteq A$ and $X \subseteq B$. Then we see that if $z \in X$, then $z \in A$ and $z \in B$, that is $z \in A \cap B$. Hence, $X \subseteq A \cap B$. Then we see that $X \in \mathcal{P}(A \cap B)$.

   **Proof of $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$**:

   Let $Y \in \mathcal{P}(A \cap B)$. Then we see that $Y \subseteq A \cap B$. Thus, if $t \in Y$, then $t \in A \cap B$. This implies $t \in A$ and $t \in B$. Thus, we see that $Y \subseteq A$ and $Y \subseteq B$. This means that $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$. Therefore $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Hence, the result follows.

3. If $A$ and $B$ are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

   **Disproof:** This statement is false. For a counterexample, we can take $A = \{1\}$ and $B = \{2\}$. Then we see that $\{1, 2\} \in \mathcal{P}(A \cup B)$ but $\mathcal{P}(A) \cup \mathcal{P}(B) = \emptyset \neq \{1\} \cup \{2\}$ which doesn’t contain the set $\{1, 2\}$.

4. If $A$, $B$ and $C$ are sets, then $A - (B \cup C) = (A - B) \cup (A - C)$.

   **Disproof:** We see that this statement is false. For a counterexample we can take $A = B = \{1\}$ and $C = \{2\}$. Then we see that $A - (B \cup C) = \emptyset \neq \{1\} = (A - B) \cup (A - C)$.

5. Suppose $A$, $B$ and $C$ are sets. If $A = B - C$, then $B = A \cup C$.

   **Disproof:** This statement is also false. For a counterexample, we can take the same sets as in the previous question, $A = B = \{1\}$ and $C = \{2\}$. Then we see that $A = B - C$, but $A \cup C = \{1, 2\} \neq \{1\} = B$.

6. Let $X$, $A$, and $B$ be sets. If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

   **Disproof:** We see that for $A = \{a\}$, $B = \{b\}$ and $X = \{a, b\}$, $X \subseteq A \cup B$ but $X \not\subseteq A$ and $X \not\subseteq B.$
7. Let $A, B, C, D$ be sets, then $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.

**Disproof:** This statement is false. For a counterexample, we can take $A = D = \mathbb{N}$ and $B = C = \mathbb{Z}$. Then we see that $(0, 0) \notin (A \cup C) \times (B \cup D)$, since $A \cup C = B \cup D = \mathbb{Z}$. But we also see that $(0, 0) \notin (A \times B)$ and $(0, 0) \notin (C \times D)$, since $0 \notin \mathbb{N}$.

8. If $A$, $B$, and $C$ are sets, then $(A \cup B) - C = (A - C) \cup (B - C)$.

**Proof:** Since this is a set equality, we need to show $(A \cup B) - C \subseteq (A - C) \cup (B - C)$ and $(A - C) \cup (B - C) \subseteq (A \cup B) - C$.

**Proof of $(A \cup B) - C \subseteq (A - C) \cup (B - C)$:** Assume that $x \in (A \cup B) - C$. Then we know that $x \in A \cup B$ and moreover $x \notin C$. That means $x \notin C$ and moreover, $x \in A$ or $x \in B$. Then we have two cases, $x \notin C$ and $x \in A$ or $x \notin C$ and $x \in B$. Since these two cases are similar, WLOG, assume $x \notin C$ and $x \in A$. Thus, $x \in (A - C)$. This implies $x \in (A - C) \cup (B - C)$. Therefore, $(A \cup B) - C \subseteq (A - C) \cup (B - C)$.

**Proof of $(A - C) \cup (B - C) \subseteq (A \cup B) - C$:** Let $y \in (A - C) \cup (B - C)$. Then we have two cases: $y \in (A - C)$ or $y \in (B - C)$. Since these cases are similar, WLOG we can assume $y \in (A - C)$. Then we see that $y \in A$ and $y \notin C$. Thus, we see $y \in A \cup B$ and $y \notin C$. Therefore $y \in (A \cup B) - C$. Hence, $(A - C) \cup (B - C) \subseteq (A \cup B) - C$.

9. If $m, n \in \mathbb{Z}$, then \( \{ x \in \mathbb{Z} : mn \mid x \} \subseteq \{ x \in \mathbb{Z} : m \mid x \} \cap \{ x \in \mathbb{Z} : n \mid x \} \).

**Proof:** Let $m, n \in \mathbb{Z}$. Assume that $s \in \{ x \in \mathbb{Z} : mn \mid x \}$. Then we see that $mn \mid s$ and thus, $s = mnk$ for some $k \in \mathbb{Z}$. Hence, we see $s = m(nk)$ and $s = n(mk)$. Since $nk, mk \in \mathbb{Z}$, we get that $n \mid s$ and $m \mid s$. Thus, $s \in \{ x \in \mathbb{Z} : n \mid x \}$ and $s \in \{ x \in \mathbb{Z} : m \mid x \}$. This implies, $s \in \{ x \in \mathbb{Z} : m \mid x \} \cap \{ x \in \mathbb{Z} : n \mid x \}$. Therefore $\{ x \in \mathbb{Z} : mn \mid x \} \subseteq \{ x \in \mathbb{Z} : m \mid x \} \cap \{ x \in \mathbb{Z} : n \mid x \}$.

10. If $m, n \in \mathbb{Z}$, then $\{ x \in \mathbb{Z} : mn \mid x \} = \{ x \in \mathbb{Z} : m \mid x \} \cap \{ x \in \mathbb{Z} : n \mid x \}$.

**Disproof:** This statement is false. We see that if we take $m = n = 2$, then
\[ \{ x \in \mathbb{Z} : mn \mid x \} = \{ x \in \mathbb{Z} : 4 \mid x \} , \text{ and } \{ x \in \mathbb{Z} : m \mid x \} \cap \{ x \in \mathbb{Z} : n \mid x \} = \{ x \in \mathbb{Z} : 2 \mid x \} . \] Thus, since $2 \in \{ x \in \mathbb{Z} : 2 \mid x \}$, but $2 \notin \{ x \in \mathbb{Z} : 4 \mid x \}$, we see that the statement is false.

11. Suppose $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ satisfying, $x, y > 0$ and $x^k = y$. Then prove that $\{ x^a : a \in \mathbb{Q} \} = \{ y^a : a \in \mathbb{Q} \}$.

**Proof:** This is a set equality. Thus, we need to prove $\{ x^a : a \in \mathbb{Q} \} \subseteq \{ y^a : a \in \mathbb{Q} \}$ and $\{ y^a : a \in \mathbb{Q} \} \subseteq \{ x^a : a \in \mathbb{Q} \}$.

**Proof of $\{ x^a : a \in \mathbb{Q} \} \subseteq \{ y^a : a \in \mathbb{Q} \}$:** Let $z \in \{ x^a : a \in \mathbb{Q} \}$. Then we know that $z = x^a$ for some $a \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ we know that $a = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then, since $x^k = y$, we see that $z = x^{\frac{p}{q}} = y^{\frac{p}{qk}}$. Thus, we see that $z \in \{ y^a : a \in \mathbb{Q} \}$ since $\frac{p}{qk} \in \mathbb{Q}$.

**Proof of $\{ y^a : a \in \mathbb{Q} \} \subseteq \{ x^a : a \in \mathbb{Q} \}$:** Let $u \in \{ y^a : a \in \mathbb{Q} \}$. Then we see that $u = y^a$ for some $a \in \mathbb{Q}$. Since $a \in \mathbb{Q}$ we know that $a = \frac{s}{t}$ for some $s \in \mathbb{Z}$ and $t \in \mathbb{N}$. Then, since $x^k = y$, we see that $z = y^{\frac{s}{t}} = x^{\frac{sk}{t}}$. Thus, we see that $z \in \{ x^a : a \in \mathbb{Q} \}$ since $\frac{sk}{t} \in \mathbb{Q}$.