Homework 5

1. Prove that, $\forall n \in \mathbb{N}, \sum_{k=1}^{n} k^3 = \left( \sum_{k=1}^{n} k \right)^2$.

**Proof**: We are going to use mathematical induction.

**Base Case**: We see that for $n = 1$, the statement is $\sum_{k=1}^{1} k^3 = 1 = \left( \sum_{k=1}^{1} k \right)^2$. Thus, the statement is true for $n = 1$.

**Inductive Step**: Let $m \geq 1$ and assume that the statement is true for $n = m$, that is, assume that $\sum_{k=1}^{m} k^3 = \left( \sum_{k=1}^{m} k \right)^2$. Then,

$$\sum_{k=1}^{m+1} k^3 = \sum_{k=1}^{m} k^3 + (m + 1)^3 = \left( \sum_{k=1}^{m} k \right)^2 + (m + 1)^3.$$

Moreover, we know that $\left( \sum_{k=1}^{m} k \right) = \frac{m(m+1)}{2}$. Hence,

$$\sum_{k=1}^{m+1} k^3 = \frac{m^2(m+1)^2}{4} + (m + 1)^3 = \frac{(m+1)^2(m^2 + 4m + 4)}{4} = \frac{(m+1)^2(m+2)^2}{4} = \left( \sum_{k=1}^{m+1} k \right)^2.$$

Therefore the statement is true for $n = m + 1$, and hence, by induction, it is true for all $n \in \mathbb{N}$.

2. Let $n \in \mathbb{N}$. Prove that $\forall n \geq 7$, $n! > 3^n$.

**Proof**: We are going to use mathematical induction.

**Base Case**: We see that for $n = 7$, the statement is $7! > 3^7$. We also know that $7! = 5040$ and $3^7 = 2187$. Thus, the statement is true for $n = 7$.

**Inductive Step**: Let $k \geq 7$, and assume that the statement is true for $n = k$, that is, $k! > 3^k$. Then we see that

$$(k+1)! = (k+1)k! > (k+1)3^k > (3)3^k = 3^{k+1},$$

since $(k+1) > 3$.

Therefore the statement is true for $n = k + 1$, and hence, by induction, it is true for all $n \in \mathbb{N}$.

3. Let $n \in \mathbb{N}$. Prove by induction on $n$, that $\exists x, y, z \in \mathbb{Z}$ such that $x \geq 2, y \geq 2$, and $z \geq 2$ and satisfy $x^2 + y^2 = z^{2n+1}$.

**Proof**: We are going to use mathematical induction.

**Base Case**: We see that for $n = 1$, the statement is $\exists x, y, z \in \mathbb{Z}$ such that $x \geq 2, y \geq 2$, and $z \geq 2$ and satisfy $x^2 + y^2 = z^3$. We see that for $x = 2, y = 2, z = 2$, this statement is true.

**Inductive Step**: Let $k \geq 1$, and assume that the statement is true for $n = k$, that is, $\exists x, y, z \in \mathbb{Z}$ such that $x \geq 2, y \geq 2$, and $z \geq 2$ and satisfy $x^2 + y^2 = z^{2k+1}$. Then we see that if we multiply the equation by $z^2$, we get $(xz)^2 + (yz)^2 = z^{2k+3} = z^{2(k+1)+1}$. Moreover, we see that $(xz), (xy) \geq 2$ since $x, z, y \geq 2$.

Therefore the statement is true for $n = k + 1$, and hence, by induction, it is true for all $n \in \mathbb{N}$.
4. Prove, using induction, that \(\forall n \in \mathbb{N}, 3 \mid (n^3 - n)\).

**Proof:** We are going to use mathematical induction.

**Base Case:** We see that for \(n = 1\), the statement is \(3 \mid (1^3 - 1) = 0\). Hence, we see that this statement is true for \(n = 1\).

**Inductive Step:** Let \(k \geq 1\), and assume that the statement is true for \(n = k\), that is, \(3 \mid (k^3 - k)\). Then we see that \(k^3 - k = 3m\) for some \(m \in \mathbb{Z}\). Thus,

\[
(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1 = (k^3 - k) + 3(k^2 + k) = 3(m + k^2 + k),
\]

and since \((m + k^2 + k) \in \mathbb{Z}\), we see \(3 \mid ((k + 1)^3 - (k + 1))\).

Therefore the statement is true for \(n = k + 1\), and hence, by induction, it is true for all \(n \in \mathbb{N}\).

5. The Fibonacci numbers are defined by the recurrence

\[
F_1 = 1 \quad F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n > 2.
\]

Show that for every \(k \in \mathbb{N}\), \(F_{4k}\) is a multiple of 3.

**Proof:** We are going to use mathematical induction.

**Base Case:** We see that for \(n = 1\), the statement is \(“F_4\) is a multiple of 3”\). We also know that \(F_3 = F_2 + F_1 = 2\), and \(F_4 = F_3 + F_2 = 3\). Hence, the statement is true for \(n = 1\).

**Inductive Step:** Let \(m \geq 1\), and assume that the statement is true for \(k = m\), that is, \(F_{4m}\) is a multiple of 3. Then we see that \(F_{4m} = 3a\) for some \(a \in \mathbb{Z}\). Thus,

\[
F_{4(m+1)} = F_{4m+4} = F_{4m+3} + F_{4m+2} = F_{4m+2} + F_{4m+1} + F_{4m+1} + F_{4m} = F_{4m+1} + 2F_{4m+1} + F_{4m} = 2F_{4m} + 3F_{4m+1},
\]

and since \((2a + F_{4m+1}) \in \mathbb{Z}\), we see \(3 \mid F_{4(m+1)}\).

Therefore the statement is true for \(k = m + 1\), and hence, by induction, it is true for all \(k \in \mathbb{N}\).

6. Let \(f(x) = x \ln x, \, x > 0\) and \(n \in \mathbb{N}\). Let \(f^{(n)}(x)\) denote the \(n\)th derivative of \(f(x)\). Prove that if \(n \geq 3\), then

\[
f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}.
\]

**Proof:** We are going to use mathematical induction.

**Base Case:** We see that for \(n = 3\), the statement is \(“f^{(3)}(x) = -\frac{1}{x^2}”\). We see that \(f'(x) = \ln(x) + 1, f''(x) = \frac{1}{x}\), and thus, \(f'(3)(x) = -\frac{1}{x^2}\). Hence, the statement is true for \(n = 3\).

**Inductive Step:** Let \(m \geq 3\), and assume that the statement is true for \(n = m\), that is, \(f^{(m)}(x) = (-1)^m \frac{(m-2)!}{x^{m-1}}\). Thus,

\[
f^{(m+1)}(x) = \frac{df^{(m)}}{dx}(x) = (-1)^m (1 - m) \frac{(m-2)!}{x^m} = (-1)^{m+1} (m - 1) \frac{(m-2)!}{x^m} = (-1)^{m+1} \frac{(m-1)!}{x^m}.
\]

Therefore the statement is true for \(n = m + 1\), and hence, by induction, it is true for all \(n \in \mathbb{N}\).

7. **Theorem:** A statement of the form \(\forall n \in \mathbb{N}; P(n)\)” is true if

- The statement \(P(1)\) is true,

and,
• given $k \geq 1$, $P(1) \land P(2) \land P(3) \land \ldots \land P(k) \implies P(k+1)$.

This procedure is called the strong induction.

Use strong induction to prove the following statement: Suppose you begin with a pile of $n$ stones ($n \geq 2$) and split this pile into $n$ separate piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have $p$ and $q$ stones in them, respectively, you compute $pq$. Show that no matter how you split the piles (eventually into $n$ piles of one stone each), the sum of the products computed at each step equals $\frac{n(n-1)}{2}$.

For example — say with start with 5 stones and split them as follows:

(5) $\rightarrow$ (3)(2) $\rightarrow$ (2)(1)(1) $\rightarrow$ (1)(1)(1)(1)(1).

Then, we get, $6 + 2 + 1 + 1 = 10 = \frac{5 \cdot 4}{2} \checkmark$.

Proof: We are going to use mathematical induction.

Base Case: We see that for $n = 2$, we have only one way of splitting them, by splitting them into two piles of one stone each. Thus, our number becomes $1 \times 1 = 1$ and we also have $\frac{2(2-1)}{2} = 1$. Hence, the statement is true for $n = 2$.

Inductive Step: Let $m \geq 2$, and assume that the statement is true for all $k \leq m$. Now, assume we have $m+1$ stones. Then for the first splitting, we have two cases.

Case 1: Splitting into two piles of 1 and $m$ stones: In this case, we have our first number to be $m \times 1 = m$. Moreover, from the inductive hypothesis, we see that if we keep splitting the pile of $m$ stones we get the number $\frac{m(m-1)}{2}$. Thus, our final number is $m + \frac{m(m-1)}{2} = \frac{(m+1)m}{2}$.

Therefore, in this case, the statement is true for $n = m+1$.

Case 2: Splitting into two piles of $p$ and $q$ stones, $p,q > 1$: In this case, we have our first number to be $pq$. Moreover, since $p,q > 1$, we have $p,q < m$. Thus, by inductive hypothesis, the number we should get by splitting the two piles into smaller piles are $\frac{p(p-1)}{2}$ and $\frac{q(q-1)}{2}$. We also know that $q = (m+1) - p$, and hence, our final number is

$$\frac{p(p-1)}{2} + \frac{q(q-1)}{2} + pq = \frac{p^2 - p + q^2 - q}{2} + pq = \frac{p^2 + q^2 - (p + q) + 2pq}{2},$$

which implies

$$\frac{(p+q)^2 - (p+q)}{2} = \frac{(m+1)^2 - (m+1)}{2} = \frac{(m+1)m}{2}.$$

Therefore, in this case, the statement is true for $n = m+1$.

Hence, we conclude that the statement is true for all $n \in \mathbb{N}$.

8. Let $F_n$ be the Fibonacci sequence defined as above. Using strong induction, show that

$$F_n = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).$$

Proof: We are going to use mathematical induction, again.
Base Case: We see that for \( n = 1 \), the statement is \( F_1 = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^1 - \left( \frac{1 - \sqrt{5}}{2} \right)^1 \right) = 1 \). Hence, we see that this statement is true for \( n = 1 \). Moreover, for \( n = 2 \), the statement becomes \( F_2 = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^2 - \left( \frac{1 - \sqrt{5}}{2} \right)^2 \right) \). Hence, we see that the statement is true for \( n = 2 \), since 

\[
\frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^2 - \left( \frac{1 - \sqrt{5}}{2} \right)^2 \right) = 1.
\]

Inductive Step: Let \( m \geq 2 \), and assume that the statement is true for all \( k \leq m \). Then we see,

\[
F_{m+1} = F_m + F_{m-1} \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^m - \left( \frac{1 - \sqrt{5}}{2} \right)^m \right) + \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{m-1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{m-1} \right) \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^m - \left( \frac{1 - \sqrt{5}}{2} \right)^m \right) + \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{m-1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{m-1} \right) \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^m + \left( \frac{1 + \sqrt{5}}{2} \right)^{m-1} \right) - \left( \left( \frac{1 - \sqrt{5}}{2} \right)^m + \left( \frac{1 - \sqrt{5}}{2} \right)^{m-1} \right) \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{m-1} \left( \left( \frac{1 + \sqrt{5}}{2} \right) + 1 \right) \right) - \left( \left( \frac{1 - \sqrt{5}}{2} \right)^{m-1} \left( \left( \frac{1 - \sqrt{5}}{2} \right) + 1 \right) \right) \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{m-1} \left( \frac{3 + \sqrt{5}}{2} \right) \right) - \left( \left( \frac{1 - \sqrt{5}}{2} \right)^{m-1} \left( \frac{3 - \sqrt{5}}{2} \right) \right) \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{m-1} \left( 6 + 2\sqrt{5} \right) \right) - \left( \left( \frac{1 - \sqrt{5}}{2} \right)^{m-1} \left( 6 - 2\sqrt{5} \right) \right) \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{m-1} \left( \frac{1 + \sqrt{5}}{2} \right)^2 \right) - \left( \left( \frac{1 - \sqrt{5}}{2} \right)^{m-1} \left( \frac{1 - \sqrt{5}}{2} \right)^2 \right) \\
= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^{m+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{m+1} \right).
\]

Therefore the statement is true for \( n = m + 1 \), and hence, by induction, it is true for all \( n \in \mathbb{N} \).