Homework 1

1. Write the following sets by listing their elements.

(a) \( A_1 = \{ x \in \mathbb{N} : x^2 < 2 \} \).
(b) \( A_2 = \{ x \in \mathbb{Z} : x^2 < 2 \} \).
(c) \( A_3 = \{ x \in \mathbb{N} : (3 \mid x) \land (x \mid 216) \} \).
(d) \( A_4 = \left\{ x \in \mathbb{Z} : \frac{x+2}{5} \in \mathbb{Z} \right\} \).
(e) \( A_5 = \{ a \in B : 6 \leq 4a + 1 < 17 \} \), where \( B = \{ 1, 2, 3, 4, 5, 6 \} \).
(f) \( A_6 = \{ x \in B : 50 < xd < 100 \text{ for some } d \in D \} \), where \( B = \{ 2, 3, 5, 7, 11, 13, \ldots \} \) and \( D = \{ 5, 10 \} \).

2. Write the following sets in set-builder notation.

(a) \( A = \{ 5, 10, 15, 20, 25, \ldots \} \).
(b) \( B = \{ \ldots, -\frac{1}{2}, 0, \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \ldots \} \).
(c) \( C = \{ 2, 4, 16, 256, 65536, \ldots \} \).
(d) \( D = \{ 2, 3, 4, 6, 8, 9, 12, 16, 18, 24 \ldots \} \).

3. Write the following sentences in symbolic logic notation and determine whether they are true or false. Make sure to note which statements/open sentences are denoted with which letter.

Example: The sentence, “The car is red and blue but not green” can be written as \((P \land Q) \land (\sim R)\) (or \((P \land Q) \land (\sim R))\), where \( P \): “The car is red”, \( Q \): “The car is blue”, and \( R \): “The car is green”. Also, the truth value of this sentence depends on the car, so it is an open sentence, not a statement.

(a) 8 is even and 5 is prime.
(b) If \( n \) is a multiple of 4 and 5, then it is a multiple of 10.
(c) \( 3 \leq x \leq 6 \).
(d) A real number \( x \) is less than \(-2\) or greater than \(2\) if its square is greater than \(4\).
(e) If a function \( f \) is differentiable everywhere then whenever \( x \in \mathbb{R} \) is a local maximum of \( f \) we have \( f'(x) = 0 \).

4. Prove the following statements.

(a) If \( n \) is even then \( n^2 + 3n + 5 \) is odd.
(b) The product of two odd numbers is odd.
(c) Let \( n, a, b, x, y \in \mathbb{Z} \). If \( n \mid a \) and \( n \mid b \), then \( n \mid (ax + by) \).
(d) Let \( n \in \mathbb{Z} \). If \( 3 \mid (n - 4) \), then \( 3 \mid (n^2 - 1) \).
(e) Let \( a \in \mathbb{Z} \). If \( 3 \mid a \) and \( 2 \mid a \), then \( 6 \mid a \).

(continued)
5. Recall that *The Mean Value Theorem* states that if we have a function $f$ which is continuous in the interval $[a, b]$ and differentiable in $(a, b)$, then whenever $x_1, x_2 \in [a, b]$ and $x_1 \neq x_2$, we have a point $c \in (x_1, x_2)$ such that $f(x_2) = f(x_1) + f'(c)(x_2 - x_1)$.

We also say that a function, $g$, is *increasing* if $g(x_1) \leq g(x_2)$ whenever $x_1 \leq x_2$.

Use The Mean Value Theorem to show that if $f$ is a differentiable function and the derivative of $f$ is positive everywhere, then $f$ is an increasing function.

6. **Definition**: We call a number $n$ an *integer root* if $n^k = m$ for some $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Use this definition to show that if $a$ and $b$ are integer roots, then so is $ab$. 