Hammack, Section 11.1, #2

The set \( R = \{(a, b), (a, c), (c, c), (b, b), (c, b), (b, c)\} \) is a relation on the set \( A = \{a, b, c\} \). This relation is not reflexive, because \( a \in A \) and \( (a, a) \notin R \). It is also not symmetric, since \( (a, b) \in R \) but \( (b, a) \notin R \) (or, similarly, since \( (a, c) \in R \) but \( (c, a) \notin R \)). The relation \( R \) is transitive; checking this by hand is somewhat tedious, although it becomes a bit easier if we notice that \( R = \{(x, y) \in A \times A : y \neq a\} \).

Hammack, Section 11.1, #8

Since \( x \) and \( y \) are integers, so is \( |x - y| \); therefore \( |x - y| < 1 \) if and only if \( |x - y| = 0 \) (there are no positive integers less than 1), which is equivalent to \( x = y \). Therefore the given relation is simply the familiar equality relation on \( \mathbb{Z} \). As we have seen, this relation is indeed reflexive, symmetric, and transitive.

Hammack, Section 11.1, #10

The relation \( R = \emptyset \) is not a reflexive relation on \( A \): if we choose some \( a \in A \) (which is possible since \( A \) is nonempty), it is clear that \( (a, a) \notin R \). On the other hand, \( R \) is symmetric: we have to verify the statement \( \text{“if } (x, y) \in R \text{ then } (y, x) \in R^\prime \text{”} \), but that implication is vacuously true (its hypothesis is always false). For similar reasons, \( R \) is also transitive, since the corresponding implication \( \text{“if } (x, y) \in R \text{ and } (y, z) \in R \text{ then } (x, z) \in R^\prime \text{”} \) is vacuously true.

Hammack, Section 11.2, #10

We need to prove that \( R \cap S \) is reflexive, symmetric, and transitive.

- Reflexive: let \( a \in A \). Since \( R \) and \( S \) are both equivalence relations and hence reflexive, \( (a, a) \) is an element of both \( R \) and \( S \), and therefore \( (a, a) \in R \cap S \).
- Symmetric: suppose \( (x, y) \in R \cap S \), so that \( (x, y) \in R \) and \( (x, y) \in S \) by definition. Since \( R \) and \( S \) are both equivalence relations and hence symmetric, \( (y, x) \) is an element of both \( R \) and \( S \), and therefore \( (x, y) \in R \cap S \).
- Transitive: suppose \( (x, y) \in R \cap S \) and \( (y, z) \in R \cap S \), so that \( (x, y) \in R \) and \( (x, y) \in S \) and \( (y, z) \in R \) and \( (y, z) \in S \). Since \( R \) and \( S \) are both equivalence relations and hence transitive, \( (x, z) \) is an element of both \( R \) and \( S \), and therefore \( (x, z) \in R \cap S \).

Hammack, Section 11.2, #12

The given statement is false. If \( R \) and \( S \) are equivalence relations, then one can show that \( R \cup S \) is both reflexive and symmetric (a good exercise!). However, we can disprove the given statement with an example of equivalence relations \( R \) and \( S \) such that \( R \cup S \) is not transitive. Given the set \( A = \{a, b, c\} \), define

\[
R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\} \quad \text{and} \quad S = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}.
\]

We can check that both \( R \) and \( S \) are equivalence relations. However, \( R \cup S \) contains both \( (a, b) \) and \( (b, c) \) but not \( (a, c) \); therefore \( R \cup S \) is not transitive and hence not an equivalence relation.
Let $P$ be a partition of the set $A$, and define

$$R = \{(x, y) \in A \times A : \exists X \in P, (x \in X \land y \in X)\}.$$ 

We first need to show that $R$ is reflexive, symmetric, and transitive.

- **Reflexive:** let $a \in A$. Since $P$ is a partition of $A$, we know that $a \in X$ for some $X \in P$ (here we use the fact that every element of $A$ is an element of some set in the partition $P$); therefore $(a, a) \in R$ (since $a \in X$ and $a \in X$).

- **Symmetric:** suppose that $(a, b) \in R$. By definition, that means that there exists $X \in P$ such that $a \in X$ and $b \in X$. Consequently, $(b, a) \in R$ (since $b \in X$ and $a \in X$).

- **Transitive:** suppose that $(a, b) \in R$ and $(b, c) \in R$. By definition, this means that there exists $X \in P$ such that $a \in X$ and $b \in X$, and there exists $Y \in P$ such that $b \in Y$ and $c \in Y$. Now the fact that $b \in X$ and $b \in Y$ implies that $X = Y$ (here we use the fact that every element of $A$ is an element of a unique set in the partition $P$); in particular, $c \in X$. Consequently, $(a, c) \in R$ (since $a \in X$ and $c \in X$).

Next we show that every element of $P$ is an equivalence class for the relation $R$. Let $X \in P$; in particular, $X \neq \emptyset$ by the definition of a partition. Choose $x \in X$, so that $x \in A$ as well. Under the relation $R$, the equivalence class of $x$ is $[x] = \{y \in A : xRy\}$, or $[x] = \{y \in A : \exists Y \in P, (x \in Y \land y \in Y)\}$. But the only $Y \in P$ such that $x \in Y$ is $X = Y$ (since $x \in X$ and every element of $A$ is an element of a unique set in $P$). Therefore $[x] = \{y \in A : y \in X\} = X$, as needed.

Finally, we show that every equivalence class for $R$ is an element of $P$. Let $[x]$ be an equivalence class for $R$, so that $x \in A$ and $xRy$ if and only if $\exists Y \in P, (x \in Y \land y \in Y)$. There exists a unique $X \in P$ such that $x \in X$. In particular, for any $y \in A$, we see that $xRy$ if and only if $y \in X$. In other words, $[x] = \{y \in A : xRy\} = \{y \in A : y \in X\} = X$, which is an element of $P$ as required.
1. Let \( R \) be a symmetric and transitive relation on a set \( A \). (These assumptions apply to both parts (a) and (b) of this problem.)
   
   (a) Show that \( R \) is not necessarily reflexive.
   
   (b) Suppose that for every \( a \in A \), there exists \( b \in A \) such that \( aRb \). Prove that \( R \) is reflexive.

   (a) Let \( A = \{1, 2\} \) and \( R = \{(1, 1)\} \). It is easy to check that \( R \) is both symmetric and transitive, but it is not reflexive because \( 2 \in A \) but \( 2 \not\in R \).

   [This simple answer is complete, but it really hides the reason why this question is in the homework assignment. One idea for creating a symmetric, transitive relation that is not reflexive is to “take away” elements being related to themselves. For example, we could consider the relation \( \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m \neq n\} \), which is certainly not reflexive (and easily verified as symmetric). However, this relation is not transitive! Transitive would mean that if \( \ell \neq m \) and \( m \neq n \), then \( \ell \neq n \) ... but this is false if \( \ell = n \). So the point is, the \( x, y, z \) in the definition of “transitive” don’t have to be distinct.]

   (b) Let \( a \in A \); we need to show that \( aRa \). By assumption, choose \( b \in A \) such that \( aRb \). By symmetry, we also have \( bRa \). By transitivity, since \( aRb \) and \( bRa \), we conclude that \( aRa \) as needed.

2. Let \( R \) be a relation on a set \( A \). Then \( \overline{R} = (A \times A) - R \) is also a relation on \( A \). Prove or disprove each of the following statements:

   (a) If \( R \) is reflexive, then \( \overline{R} \) is reflexive.
   
   (b) If \( R \) is symmetric, then \( \overline{R} \) is symmetric.
   
   (c) If \( R \) is transitive, then \( \overline{R} \) is transitive.

   (a) This statement is false for any nonempty set \( A \). Choose \( a \in A \); then \((a, a) \in R \) since \( R \) is reflexive. By the definition of complement, \((a, a) \notin \overline{R} \), which shows that \( \overline{R} \) is not reflexive. (It certainly suffices to give a similar argument for a specific nonempty set \( A \) such as \( A = \{1\} \).)

   (b) This statement is true. Suppose that \((a, b) \in \overline{R} \); by definition, this means \((a, b) \notin R \). Note that this implies that \((b, a) \notin R \) as well: if \((b, a) \) were an element of \( R \), then since \( R \) is symmetric, \((a, b) \) would be an element of \( R \) as well, a contradiction. Since \((b, a) \notin R \), we conclude that \((b, a) \in \overline{R} \). Therefore \( \overline{R} \) is symmetric.

   (c) This statement can be false. For example, take \( A = \{1, 2\} \) and \( R = \{(1, 1), (2, 2)\} \), which is transitive. Then \( \overline{R} = \{(1, 2), (2, 1)\} \); however, \((1, 2) \notin \overline{R} \) and \((2, 1) \notin \overline{R} \) but \((1, 1) \notin \overline{R} \), which shows that \( \overline{R} \) is not transitive.