Definition: A set $A$ is finite if there exists a nonnegative integer $c$ such that there exists a bijection from $A$ to $\{n \in \mathbb{N} : n \leq c\}$. (The integer $c$ is called the cardinality of $A$.)

I. (a) Let $A$ be a finite set, and let $B$ be a subset of $A$. Prove that $B$ is finite. (Hint: induction on $|A|$. Note that our proof can’t use induction on $|B|$, or indeed refer to “the number of elements in $B$” at all, because we don’t yet know that $B$ is finite!)

(b) Prove that the union of two disjoint finite sets is finite.

(c) Prove that the union of any two finite sets is finite. (Hint: $A \cup B = A \cup (B - A)$.)

(a) We proceed by induction on the nonnegative integer $c$ in the definition that $A$ is finite (the cardinality of $c$).

Basis step: $c = 0$. Then there is a bijection from $A$ to $\{n \in \mathbb{N} : n \leq 0\} = \emptyset$, and thus $A = \emptyset$ (and, for that matter, the bijection is also the empty function). This forces $B = \emptyset$ as well (since that is the only subset of $\emptyset$), which is certainly finite.

Induction step: Let $c \geq 0$. The induction hypothesis is: if $A$ has cardinality $c$, then any subset of $A$ is finite. Now suppose $A$ has cardinality $c + 1$, and let $f : A \to \{n \in \mathbb{N} : n \leq c + 1\}$ be a bijection. Define $A_1 = f^{-1}(\{1, \ldots, c\})$, and note that $f$ restricted to $A_1$ is a bijection from $A_1$ to $\{1, \ldots, c\}$, so that $A_1$ has cardinality $c$. Finally, let $B$ be a subset of $A$.

Case 1: $f^{-1}(c + 1) \notin B$. Then $B$ is a subset of $f^{-1}(\{1, \ldots, c\}) = A_1$, which has cardinality $c$; by the induction hypothesis, $B$ is finite.

Case 2: $f^{-1}(c + 1) \in B$. Then define $B_1 = B - f^{-1}(c + 1)$. As before, $B_1$ is a subset of $A_1$ and hence finite. Let $d$ be the cardinality of $B_1$ and let $g_1 : B_1 \to \{n \in \mathbb{N} : n \leq d\}$ be a bijection. Now define a function $g : B \to \{n \in \mathbb{N} : n \leq d + 1\}$ by

$$
g(x) = \begin{cases} 
g_1(x), & \text{if } x \in B_1, \\
d + 1, & \text{if } x = f^{-1}(c + 1).
\end{cases}
$$

It is not hard to check that $g$ is a bijection from $B$ to $\{n \in \mathbb{N} : n \leq d + 1\}$. In particular, $B$ is finite.

(b) Let $C$ and $D$ be finite sets with $C \cap D = \emptyset$, and let $f : C \to \{n \in \mathbb{N} : n \leq c\}$ and $g : D \to \{n \in \mathbb{N} : n \leq d\}$ be bijections. Define $h : C \cup D \to \{n \in \mathbb{N} : n \leq c + d\}$ by

$$
h(x) = \begin{cases} f(x), & \text{if } x \in C, \\
g(x) + c, & \text{if } x \in D.
\end{cases}
$$

It is (somewhat tedious but) not hard to check that $h$ is a bijection from $C \cup D$ to $\{n \in \mathbb{N} : n \leq c + d\}$. (The fact that $C \cap D = \emptyset$ is necessary to show that $h$ is a well-defined function.) In particular, $C \cup D$ is finite.

(c) Let $A$ and $B$ be finite sets. Then $B - A$ is a subset of the finite set $B$ and hence is itself finite by part (a). Consequently, since $A$ and $B - A$ are always disjoint, the set $A \cup B = A \cup (B - A)$ is the union of two disjoint finite sets and is therefore finite by part (b).
II. Let $A$ and $B$ be nonempty sets. Prove that there exists an injective function $f : A \to B$ if and only if there exists a surjective function $g : B \to A$.

First assume that $f : A \to B$ is injective. Let $D = f(A)$ be the range of $A$; then $f$ is a bijection from $A$ to $D$. Choose any $a \in A$ (possible since $A$ is nonempty). Define $g : B \to A$ by

$$g(y) = \begin{cases} f^{-1}(y), & \text{if } y \in D, \\ a, & \text{if } y \in B - D. \end{cases}$$

It is not hard to show that $g$ is a well-defined function that is surjective.

Conversely, assume that $g : B \to A$ is surjective. Define a function $f : A \to B$ as follows: for each $a \in A$, choose $b_a \in B$ such that $g(b_a) = a$, and define $f(a) = b_a$. It is not hard to check that $f$ is a well-defined function that is injective. [Interested students who just read this proof might wish to find some introductory information about something called the “Axiom of Choice”.

Section 13.1

Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

2. $\mathbb{R}$ and $(\sqrt{2}, \infty)$
4. The set of even integers and the set of odd integers
8. $\mathbb{Z}$ and $S = \{x \in \mathbb{R} : \sin x = 1\}$
10. $\{0, 1\} \times \mathbb{N}$ and $\mathbb{Z}$
14. $\mathbb{N} \times \mathbb{N}$ and $\{(n, m) \in \mathbb{N} \times \mathbb{N} : n \leq m\}$. (Hint: draw “graphs” of both sets. The northwest–southeast diagonal slices of the first set look a lot like the horizontal slices of the second set . . .\)

2. Adapting one of the bijections from the book, define $f : \mathbb{R} \to (\sqrt{2}, \infty)$ by $f(x) = e^x + \sqrt{2}$. It’s easy to check that $f$ is a well-defined and bijective function (for example, $f^{-1}(y) = \ln(y - \sqrt{2})$ is its inverse function).
4. A bijection $f : \{\text{even integers}\} \to \{\text{odd integers}\}$ is $f(n) = n + 1$; or $f(n) = n - 1$; or indeed $f(n) = n + k$ or $f(n) = n - k$ for any fixed odd integer $k$.
8. Note that $S = \{x \in \mathbb{R} : \sin x = 1\} = \{\ldots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots\}$. We see that one bijection $f : \mathbb{Z} \to S$ is $f(n) = 2\pi n + \frac{\pi}{2}$.
10. Define a function $f : \{0, 1\} \times \mathbb{N} \to \mathbb{Z}$ by

$$f((b, n)) = \begin{cases} n, & \text{if } b = 0, \\ 1 - n, & \text{if } b = 1. \end{cases}$$

It is not hard to show that $f$ is a bijection; for example, its inverse function is

$$f^{-1}(m) = \begin{cases} (0, m), & \text{if } m \geq 1, \\ (1, 1 - m), & \text{if } m \leq 0. \end{cases}$$

14. Inspired by the given hint, define a function $f : \{(n, m) \in \mathbb{N} \times \mathbb{N} : n \leq m\} \to \mathbb{N} \times \mathbb{N}$ by $f((n, m)) = (n, m + 1 - n)$. Since $n \leq m$ for elements of the domain, we see that $m + 1 - n \geq 1$, and so the values really do lie in the codomain. One can check that this function is a bijection; for example, its inverse function is $f^{-1}((x, y)) = (x, x + y - 1)$. 

4. The set of even integers and the set of odd integers
III. Suppose that two sets $A$ and $B$ have the same cardinality. Prove that $\mathcal{P}(A)$ and $\mathcal{P}(B)$ have the same cardinality as each other.

Let $f: A \to B$ be a bijection. Define a function $g: \mathcal{P}(A) \to \mathcal{P}(B)$ by $g(X) = f(X)$ for any $X \in \mathcal{P}(A)$. (Let’s interpret the notation carefully: $g(X)$ is a function value since $X$ is an element of the domain $\mathcal{P}(A)$ of $g$, while $f(X)$ is an image since $X$ is a subset of the domain $A$ of $f$.) We claim that $g$ is a bijection.

First, let $Y \in \mathcal{P}(B)$. Then if we set $X = f^{-1}(Y)$, then $g(X) = f(X) = f(f^{-1}(Y)) = Y$. (The identity $f(f^{-1}(Y)) = Y$ isn’t true in general, but it is true when $f$ is surjective, as shown on the previous homework.) In particular, $g$ is surjective.

Next, let $X_1$ and $X_2$ be elements of $\mathcal{P}(A)$ and suppose that $g(X_1) = g(X_2)$. Then $f(X_1) = f(X_2)$ (as images). Taking preimages of both sets yields $f^{-1}(f(X_1)) = f^{-1}(f(X_2))$. Since $f$ is injective, we have $f^{-1}(f(X_1)) = X_1$ and $f^{-1}(f(X_2)) = X_2$ as shown on the previous homework; therefore $X_1 = X_2$. In particular, $h$ is surjective.

Section 13.2

2. Prove that the set $A = \{(m, n) \in \mathbb{N} \times \mathbb{N}: m \leq n\}$ is countably infinite.

We already know that $\mathbb{N} \times \mathbb{N}$ is countably infinite. By problem 14 in Section 13.1 (above), $A$ has the same cardinality as $\mathbb{N} \times \mathbb{N}$. Therefore $A$ is also countably infinite.

4. Prove that the set of all irrational numbers is uncountable.

Let $I$ be the set of irrational numbers. Suppose for the sake of contradiction that $I$ is countably infinite. We already know that $\mathbb{Q}$ is countably infinite. Then $\mathbb{R} = \mathbb{Q} \cup I$, the union of two countably infinite sets, would be countably infinite as well; but this contradicts the known fact that $\mathbb{R}$ is uncountable. Therefore $I$ is not countably infinite; since $I$ is certainly an infinite set, we conclude that $I$ is uncountable.

6. Prove or disprove: There exists a bijective function $f: \mathbb{Q} \to \mathbb{R}$.

Disproof: if there were such a bijective function, then $\mathbb{Q}$ and $\mathbb{R}$ would have the same cardinality. But we know that $\mathbb{Q}$ is countably infinite while $\mathbb{R}$ is uncountable, and therefore they do not have the same cardinality. We conclude that there is no bijection from $\mathbb{Q}$ to $\mathbb{R}$.

8. Prove or disprove: The set $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.

Proof: we know that both $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite, and we know that the Cartesian product of two countably infinite sets is again countably infinite. Therefore $\mathbb{Z} \times \mathbb{Q}$ is countably infinite.

12. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.

We know that $\mathbb{N} \times \mathbb{N}$ is countably infinite, so let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. For each $m \in \mathbb{N}$, define $A_m = f(\{m\} \times \mathbb{N})$. The sets $\{m\} \times \mathbb{N}$ form a partition of the domain $\mathbb{N} \times \mathbb{N}$; since $f$ is a bijection, the sets $A_m$ form a partition of the codomain $\mathbb{N}$. Also, each set $\{m\} \times \mathbb{N}$ is countably infinite (there is an obvious bijection from each to $\mathbb{N}$), and therefore their images $A_m$ under the bijection $f$ are also each countably infinite. Therefore $\{A_m: m \in \mathbb{N}\}$ is the desired partition of $\mathbb{N}$; this partition has $|\mathbb{N}| = \aleph_0$ elements, as desired.

For example, if we take the bijection $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f((m, n)) = 2^{m-1}(2n - 1)$, as in problem #15 of Section 13.2, then $A_m = \{2^{m-1}(2n - 1): n \in \mathbb{N}\}$ is the set of all
positive integers that are divisible by $2^{m-1}$ but not divisible by $2^m$. For example,

\begin{align*}
A_1 &= \{1, 3, 5, 7, 9, 11, \ldots \} \\
A_2 &= \{2, 6, 10, 14, 18, 22, \ldots \} \\
A_3 &= \{4, 12, 20, 28, 36, 44, \ldots \} \\
A_4 &= \{8, 24, 40, 56, 72, 88, \ldots \} \\
&\vdots
\end{align*}

These $\{A_m : m \in \mathbb{N}\}$ form a partition of $\mathbb{N}$ into $\aleph_0$ countably infinite subsets.

14. Suppose $A = \{(m, n) \in \mathbb{N} \times \mathbb{R} : n = \pi m\}$. Is it true that $|\mathbb{N}| = |A|$?

Yes. Note that $A = \{(1, \pi), (2, 2\pi), (3, 3\pi), \ldots \}$. Define a function $f : A \to \mathbb{N}$ by $f((m, n)) = m$. It is easy to check that $f$ is a bijection. Therefore $|A| = |\mathbb{N}|$.

SECTION 13.3

4. Prove or disprove: If $A \subseteq B \subseteq C$ and $A$ and $C$ are countably infinite, then $B$ is countably infinite.

Proof: Since $A$ is infinite and $A \subseteq B$, we see that $B$ is infinite. (This is the contrapositive of the first problem of this homework.) Then $B$ is an infinite subset of the countably infinite set $C$, and therefore $B$ is itself countably infinite.

6. Prove or disprove: Every infinite set is a subset of a countably infinite set.

Disproof: consider $\mathbb{R}$, which is uncountable. If $\mathbb{R}$ were a subset of a countably infinite set, then it too would be countably infinite, which is a contradiction. Therefore $\mathbb{R}$ is an infinite set that is not a subset of any countably infinite set. (Indeed, no uncountable set is a subset of a countably infinite set.)

8. Prove or disprove: The set $\{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$ of infinite sequences of integers is countably infinite.

Disproof: let $S = \{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$, and let $f : \mathbb{N} \to S$ be a function. We claim that $f$ is not surjective. In particular, this will show that there is no bijection from $\mathbb{N}$ to $S$, and so $S$ is not countably infinite.

Let $a_i^{(j)}$ denote the $i$th element in the sequence $f(j)$. Define a sequence $(b_1, b_2, b_3, \ldots)$ where

$$b_i = \begin{cases} 1, & \text{if } a_i^{(i)} \neq 1, \\ -1, & \text{if } a_i^{(i)} = 1. \end{cases}$$

Then $(b_1, b_2, b_3, \ldots) \in S$. On the other hand, for every $i$, we see that $(b_1, b_2, b_3, \ldots) \neq f(i)$, since their $i$th coordinates $b_i$ and $a_i^{(i)}$ are different. Therefore $(b_1, b_2, b_3, \ldots)$ is not in the range of $f$, and therefore $f$ is not surjective.
9. Prove that if $A$ and $B$ are finite sets with $|A| = |B|$, then any injection $f: A \to B$ is also a surjection. Show this is not necessarily true if $A$ and $B$ are not finite.

Suppose, for the sake of contradiction, that $f$ is not surjective, and choose $b \in B$ that is not in the range of $f$. Let $z$ be an object that is not an element of $A$, define $A_1 = A \cup \{z\}$, and define a function $g: A_1 \to B$ by

$$g(x) = \begin{cases} b, & \text{if } x = z, \\ f(x), & \text{if } x \neq z. \end{cases}$$

It is easy to check that $g$ is also injective. However, $|A_1| = |A| + 1 = |B| + 1 > |B|$, which violates the Pigeonhole Principle and is thus a contradiction. Therefore $f$ must be surjective.

If $A$ and $B$ are not finite, we have counterexamples such as $A = \mathbb{N}$, $B = \mathbb{Z}$, and $f: A \to B$ being the inclusion map $f(n) = n$; then $f$ is injective but not surjective (since $-1$ is not in the range, for example).

10. Prove that if $A$ and $B$ are finite sets with $|A| = |B|$, then any surjection $f: A \to B$ is also an injection. Show this is not necessarily true if $A$ and $B$ are not finite.

Suppose, for the sake of contradiction, that $f$ is not injective, and choose $a_1, a_2 \in A$ with $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. Define $A_1 = A - \{a_2\}$, and define $g: A_1 \to B$ by $g(x) = f(x)$. It is easy to check that $g$ is also surjective. However, $|A_1| = |A| - 1 = |B| - 1 < |B|$, which violates the Pigeonhole Principle and is thus a contradiction. Therefore $f$ must be injective.

If $A$ and $B$ are not finite, we have counterexamples such as $A = \mathbb{N} \times \mathbb{N}$, $B = \mathbb{N}$, and $f((m,n)) = n$; then $f$ is surjective but not injective (since $f((2,3)) = 3 = f((7,3))$, for example).