Topics to review for Math 220 Final Exam, on December 20.

Chapter 1:

(1) The notions of a set, subset, element. Please make sure that you clearly understand the distinction between the usage of the symbols '$\in$' and '$\subseteq$', and also do not confuse things such as $\emptyset$ and $\{\emptyset\}$. In particular, please check that you understand why writing $\{1\} \subseteq \{1, 2, 3\}$ is correct, and $1 \in \{1, 2, 3\}$ is correct, but $1 \subseteq \{1, 2, 3\}$ is incorrect.
(2) The notion of a power set.
(3) Set operations: union, intersection, difference, complement. Venn diagrams.
(4) Direct product of sets.
(5) Unions and intersections (and complements) of indexed collections of sets.

Chapter 2:

(1) Statements.
(2) Logical operations – conjunction, disjunction, negation.
(3) Open sentences.
(4) Logical equivalences and truth tables. You should be able to construct truth tables for compound statements, and use them to check if two compound statements are logically equivalent.
(5) Some important logical equivalences, for example, De Morgan laws (both for statements and sets). This means, you should be able to correctly negate complex statements involving conjunctions, disjunctions and negations. Distributivity laws.
(6) The implication. Converse and contrapositive. The fact that $P \Rightarrow Q$ is logically equivalent to $\sim P \vee Q$. Logical equivalence of the implication and the contrapositive. Please note: there are 4 different things: the implication $P \Rightarrow Q$; the converse $Q \Rightarrow P$; the contrapositive $\sim Q \Rightarrow \sim P$; and the negation of the implication itself, $\sim (P \Rightarrow Q)$. Of these, only the implication and its contrapositive are logically equivalent. Otherwise, none of these implies the other. In particular, if $P \Rightarrow Q$ is true, it does not at all mean that the converse should be true (see if you can come up with examples). The negation of the implication $\sim (P \Rightarrow Q)$ is logically equivalent to $\sim (\sim P \vee Q) \equiv P \wedge \sim Q$. In particular, please do not confuse this with the contrapositive or converse, or the contrapositive to the converse (which is a very popular mistake). You should be able to come up with verbal examples, such as "if I do not buy a lottery ticket, I will not win the lottery". Try making the converse, contrapositive, and the negation of this statement, and look at the differences.
(7) The biconditional.
(8) Quantifiers.
(9) Negation of statements involving quantifiers.

Chapters 4, 5, 6, 7, 8, 9, and 10:

In these chapters, we learned various methods of proofs, including

(1) Direct proof
(2) Proof by contrapositive
(3) Proof by cases
(4) Disproving statements by a counterexample
(5) Proof by contradiction
(6) Existence (constructive or non-constructive) proofs and uniqueness proofs
(7) Mathematical Induction
    (a) Mathematical Induction
    (b) Strong Mathematical Induction
    (c) Proof by smallest counterexample

The statements you will be expected to prove (or disprove) will be either about relationships between different sets (which may involve image or preimage of functions), or about unions or intersections of indexed collections of sets, or about divisibility and congruence of integers, or about rational/irrational numbers.

For two sets A, B, you need to know how to prove (i) $A \subseteq B$, (ii) $A = B$ (proving $A \subseteq B$ and $B \subseteq A$)

You need to be able to state and use

(1) the Fundamental Theorem of Arithmetic.
(2) The Division Algorithm

and the useful results

(1) Proposition 10.1: If $p|(a_1 \cdot a_2 \cdots a_n)$, where p is prime, $a_1, \ldots, a_n, n$ are integers, and $n \geq 2$, then $p|a_i$ for at least of the a_i.
(2) Proposition 7.1: If $a, b \in \mathbb{N}$, then there exist $k, l \in \mathbb{Z}$ such that $gcd(a, b) = ak + bl$.

You also need to know how to prove that there are infinitely many prime numbers (using proof by contradiction).

Chapter 11:

(1) Relation from A to B, inverse relation
(2) Relation on A, reflexive/symmetric/transitive, equivalence relation, equivalence classes
(3) Partition of a set
(4) $\mathbb{Z}_n$: Integers modulo n, addition and multiplication on $\mathbb{Z}_n$
(5) We have proved

Theorem 11.1: Suppose that R is an equivalence relation on a set A and $a, b \in A$. Then $[a] = [b]$ if and only if aRb.

Theorem 11.2: Suppose that R is an equivalence relation no a set A. Then the set $\{[a] : a \in A\}$ of the equivalence classes of R forms a partition of A.

Chapter 12:

(1) Definition of a function $f : A \to B$, domain, codomain, and range of f, $f(C)$ (image of a set C under f for $C \subseteq A$), $f^{-1}(D)$ (preimage of a set D for $D \subseteq B$), composition of functions
(2) Injective, surjective, bijective functions
(3) Inverse function
(4) The Pigeonhole Principle

Chapter 13:

(1) Definition of two sets A, B have the same cardinality ($|A| = |B|$), or unequal cardinality ($|A| \neq |B|$)
(2) Definition of a set is: finite, infinite, countably infinite, uncountable
(3) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all countably infinite, $\mathbb{R}$ is uncountable (you need to know the idea of the diagonal procedure)
(4) We have shown
    (a) If A, B are both countably infinite, then so is $A \times B$.
    (b) If A, B are both countably infinite, then so is $A \cup B$.
    (c) An infinite subset of a countably infinite set is countably infinite. As a corollary, if $U \subseteq A$ and U is uncountable, then A is uncountable.
(5) Comparing cardinalities:
    (a) Definition of $|A| \leq |B|, |A| < |B|$
    (b) $|\mathbb{N}| < |\mathbb{R}|$
    (c) $|A| < |\mathcal{P}(A)|$ for any set A.