The Well-Ordering Principle and its applications

- **Definition.** A nonempty subset $S$ of $\mathbb{R}$ is **well-ordered** if every non-empty subset of $S$ has a smallest element.

- **The Well-Ordering Principle:** The set $\mathbb{N}$ is well-ordered.

- **Example.** The following sets are well-ordered:
  1. $\mathbb{N} \cup \{0\}$
  2. $\mathbb{N} \cup \{-1, 0\}$
  3. $\mathbb{N} \cup \{-3, -2, -1\}$
  4. $\{n \in \mathbb{N} : n > 5\}$

- **Example.** The following sets are NOT well-ordered.
  1. $\mathbb{R}$ (the open interval $(0, 2)$ is a non-empty subset of $\mathbb{R}$ but it has no smallest element)
  2. $\mathbb{Z}$ (the set of negative integers is a non-empty subset of $\mathbb{Z}$ but with no smallest element)
  3. the interval $[0, 1]$ (because $(0, 1)$ is a non-empty subset of $[0, 1]$ without smallest element)

- **Proposition.** Every non-empty subset of a well-ordered set is well-ordered.

- **Applications**
  1. The **Division Algorithm:** Given $a, b \in \mathbb{Z}$ with $b > 0$, there exist $q, r \in \mathbb{Z}$ for which $a = qb + r$ and $0 \leq r < b$.

    **Proof.** Consider the set $S = \{a - xb : x \in \mathbb{Z}, a - xb \geq 0\}$.
    
    It holds $S \subseteq \mathbb{N} \cup \{0\}$.
    
    To see $S$ is non-empty, take $x = -|a| \in \mathbb{Z}$, so
    
    $a - xb = a + |a|b \geq 0$
    
    because $b$ is a positive integer (so $b \geq 1$), hence $a + |a|b \in S$. By the fact that $\mathbb{N} \cup \{0\}$ is well-ordered, there is a least element in $S$, call it $r$, so $r \geq 0$ and $r = a - qb$ for some $q \in \mathbb{Z}$. We now prove $r < b$. If $r \geq b$, then
    
    $0 \leq r - b = (a - qb) - b = a - (q + 1)b$
    
    would imply $r - b \in S$, which contradicts that $r$ is the smallest in $S$. 


  3. **Theorem.** (Principle of Mathematical Induction) For every $n \in \mathbb{N}$ let $P(n)$ be a statement. If
    
    (a) $P(1)$ is a true statement, and
    (b) the implication $P(k) \Rightarrow P(k + 1)$ is true for all $k \in \mathbb{N}$
    
    then $P(n)$ is true for all $n \in \mathbb{N}$.

    **Proof.** We prove by contradiction. Assume that the set $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$
    
    is non-empty. Then there is a least element $l$ in $S$ by the Well-Ordering Principle. Since $P(1)$ is true by (a), we must have $l \geq 2$. So $l - 1 \in \mathbb{N}$. Since $l$ is the least element in $S$, $l - 1 \notin S$, so $P(l - 1)$ is true. Then we must have $P(l)$ is true by (b). This contradicts that $l \in S$. We conclude that $S = \emptyset$. \qed