This midterm has 6 questions on 7 pages Duration: 50 minutes

• Read all the questions carefully before starting to work.
• Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
• Continue on the back of the previous page if you run out of space.
• Attempt to answer all questions for partial credit.
• This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First): ________________________________________________________________

Student Number: ________________________________________________________________

Signature: ________________________________________________________________

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1. (a) For each statement (i), (ii), and (iii): decide whether it is true or false (write True or False in the box), and then write the negation of the statement (your answer cannot contain the $\sim$ symbol).

(i) $\forall n \in \mathbb{Z}, \exists y \in \mathbb{R} - \{0\}$ such that $y^n \leq y$. Answer True

Negation: $\exists n \in \mathbb{Z}$, such that $\forall y \in \mathbb{R} - \{0\}$, $y^n > y$.

(ii) $\exists y \in \mathbb{R} - \{0\}$ such that $\forall n \in \mathbb{Z}$, $y^n \leq y$. Answer True

Negation: $\forall y \in \mathbb{R} - \{0\}$, $\exists n \in \mathbb{Z}$ such that $y^n > y$.

(iii) $\forall x \in \mathbb{R} - \{0\}$, $x \leq \frac{1}{3}$ or $\frac{1}{x} \leq 3$. Answer True

Negation: $\exists x \in \mathbb{R} - \{0\}$, such that $x > \frac{1}{3}$ and $\frac{1}{x} > 3$.

(b) Determine, without proof, whether the following statements are true or false. Write True or False in the boxes.

(i) $(\mathbb{R} \times \mathbb{Z}) \cap (\mathbb{Z} \times \mathbb{R}) = \mathbb{Z} \times \mathbb{Z}$ Answer True

(ii) $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) = \mathbb{R} \times \mathbb{R}$ Answer False

(c) Let $A = \{\emptyset, 1, 2, \{1\}\}$ and $\mathcal{P}(A)$ be the power set of $A$. Determine whether the following statements are true or false. Write True or False in the boxes.

(i) $1 \in \mathcal{P}(A)$ Answer False

(ii) $\{1\} \in \mathcal{P}(A)$ Answer True

(iii) $\{\{1\}\} \in \mathcal{P}(A)$ Answer True

(iv) $\emptyset \in \mathcal{P}(A)$ Answer True
2. (a) (2 marks) Given that the statement $S$ is true, determine whether

$$
(P \Rightarrow (Q \lor R)) \Rightarrow (S \lor (\neg (Q \land R)))
$$

is true or false; and explain your answer.

**Solution:** We see that the statement $(S \lor (\neg (Q \land R)))$ is true given that $S$ is true. Moreover, if we have a statement of the form $A \Rightarrow B$, and that $B$ is true, we see that $A \Rightarrow B$ is true regardless $A$ is true or false. Thus we see that, since $(S \lor (\neg (Q \land R)))$ is true, $(P \Rightarrow (Q \lor R)) \Rightarrow (S \lor (\neg (Q \land R)))$ is true.

(b) (6 marks) Given that the statement $S$ is true, and that the statement

$$
(S \lor (\sim (Q \land R))) \iff (P \Rightarrow (Q \lor R))
$$

is false, determine whether the statements $P$, $Q$ and $R$ are true or false; explain your answer.

**Solution:** Since $S$ is true, any statement of the form $S \lor T$ is true. Therefore the left-hand side of the given biconditional (**) is true. But we know that the biconditional (**) itself is false; this means that its two substatements have opposite truth values. Since the left-hand side is true, the right-hand side $P \Rightarrow (Q \lor R)$ must be false. The only way an implication can be false is if its hypothesis is true but its conclusion is false. Therefore the hypothesis $P$ must be True, and the conclusion $Q \lor R$ must be false. Finally, an or-statement is false only if both its substatements are false; therefore $Q$ and $R$ are both False.
3. Prove that $\sqrt{2}$ is irrational. You may use (without needing to prove it) the fact that $n^5$ is even if and only if $n$ is even.

Solution: Proof by contradiction. Suppose that $\sqrt{2}$ is rational. By definition, we can then write $\sqrt{2} = \frac{a}{b}$ for integers $a, b$ with $b \neq 0$; moreover, we may assume $\gcd(a, b) = 1$ (so $\frac{a}{b}$ is fully reduced). Then $2 = \frac{a^2}{b^2}$, or $a^5 = 2b^5$. So $a^5$ is even, therefore $a$ is even. Now we can write $a = 2k$ for some $k \in \mathbb{Z}$. This leads to $(2k)^5 = 2b^5$, which gives $b^5 = 2(2^3k^5)$. In particular, $b^5$ is even, hence $b$ is even. But now both $a$ and $b$ are even, so $\gcd(a, b) \geq 2$, contradicting the assumption that $\gcd(a, b) = 1$. This contradiction shows that $\sqrt{2}$ is irrational. q.e.d.
4. Let \( a, b \in \mathbb{Z} \). Prove that \((a - 3)b^2\) is odd if and only if \( a \) is even and \( b \) is odd.

**Solution:** First we prove: if \( a \) is even and \( b \) is odd, then \((a - 3)b^2\) is odd. Write \( a = 2m \) and \( b = 2n + 1 \) for some integers \( m \) and \( n \). Then \((a - 3)b^2 = (2m - 3)(2n + 1)^2 = 2(4mn^2 + 4mn + m - 6n^2 - 6n - 2) + 1 \) is odd (since \( 4mn^2 + 4mn + m - 6n^2 - 6n - 2 \) is also an integer).

Next we prove: if \((a - 3)b^2\) is odd, then \( a \) is even and \( b \) is odd. We do this by proving the contrapositive: if \( a \) is odd or \( b \) is even, then \((a - 3)b^2\) is even.

Case 1: \( a \) is odd. Write \( a = 2k + 1 \) for some integer \( k \); then \((a - 3)b^2 = (2k + 1 - 3)b^2 = 2((k - 1)b^2) \) is even (since \((k - 1)b^2\) is also an integer).

Case 2: \( b \) is even. Write \( b = 2k \) for some integer \( k \); then \((a - 3)b^2 = (a - 3)(2k)^2 = 2((a - 3)2k^2) \) is even (since \((a - 3)2k^2\) is also an integer).

In either case, \((a - 3)b^2\) is even. q.e.d.
5. Prove: If \( n \equiv 1 \pmod{2} \) and \( m \equiv 3 \pmod{4} \), then \( n^2 + m \equiv 0 \pmod{4} \).

Solution: Since \( n \equiv 1 \pmod{2} \), \( 2 \mid (n - 1) \). There is \( k \in \mathbb{Z} \) such that \( n - 1 = 2k \). So \( n = 2k + 1 \). Since \( m \equiv 3 \pmod{4} \), \( 4 \mid (m - 3) \). There is \( l \in \mathbb{Z} \) such that \( m - 3 = 4l \), so \( m = 4l + 3 \). Now, \( n^2 + m = (2k + 1)^2 + (4l + 3) = 4(k^2 + k + l + 1) \). Therefore, \( 4 \mid (n^2 + m) \) as \( k^2 + k + l + 1 \in \mathbb{Z} \). So \( n^2 + m \equiv 0 \pmod{4} \). q.e.d.
6. Let \( n \in \mathbb{Z} \). Show that \( 2 \mid (n^4 - 5) \) if and only if \( 4 \mid (n^2 + 7) \). (You can use without proof the fact that \( ab \) is odd if and only if \( a \) and \( b \) are both odd.)

**Solution:** Since \( 2 \mid (n^4 - 5) \), \( n^4 - 5 = 2k \) for some \( k \in \mathbb{Z} \). So \( n^4 = 2k + 5 = 2(k+2) + 1 \) is odd. Since \( n^4 = nn^3 \), so \( n \) is odd by the Fact, and then \( n = 2l + 1 \) for some \( l \in \mathbb{Z} \).

Now \( n^2 + 7 = (2l + 1)^2 + 7 = 4(l^2 + l + 2) \), so \( 4 \mid (n^2 + 7) \) as \( l^2 + l + 2 \in \mathbb{Z} \).

Now, assume \( 4 \mid (n^2 + 7) \). Then \( n^2 + 7 = 4p \) for some \( p \in \mathbb{Z} \). So \( n^2 = 4p - 7 = 2(2p - 4) + 1 \) is odd as \( 2p - 4 \in \mathbb{Z} \). So \( n^4 = n^2 n^2 \) is odd, and \( n^4 - 5 \) is even (odd - odd is even). We have \( 2 \mid (n^4 - 5) \). q.e.d.