1. (Chapter 10: Question 8) If \( n \in \mathbb{N} \), then
\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.
\]

**Proof.** We will prove this by using induction on \( n \).

**Base step:** When \( n = 1 \) the left hand side is \( \frac{1}{2!} = \frac{1}{2} \). When \( n = 1 \) the right hand side is \( 1 - \frac{1}{(1+1)!} = \frac{1}{2} \) which proves the equality is true when \( n = 1 \).

**Inductive step:** Let \( n \geq 1 \). We will assume that \( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \). Our goal is to show that \( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!} \). Using our inductive assumption we have
\[
\left( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} \right) + \frac{n+1}{(n+2)!} = \left( 1 - \frac{1}{(n+1)!} \right) + \frac{n+1}{(n+2)!} = \left( 1 - \frac{1}{(n+1)!} \right) - \frac{n+1}{(n+1)!} \cdot \frac{1}{(n+2)!}.
\]

By induction we have thus shown that \( \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!} \) for all natural numbers \( n \). \( \square \)

2. (Chapter 10: Question 12) For any integer \( n \geq 0 \), it follows that \( 9|(4^{3n} + 8) \).

**Proof.** We will do induction on \( n \).

**Base step:** For \( n = 0 \), \( 4^{3^0} + 8 = 4^0 + 8 = 9 \) is divisible by 9.

**Inductive step:** Assume \( 9|(4^{3n} + 8) \) for \( n \geq 0 \). This means \( 4^{3n} + 8 = 9p \) for some \( p \in \mathbb{Z} \). Then
\[
4^{3(n+1)} + 8 = 4^3 \cdot 4^{3n} + 8 = 64 \cdot 4^{3n} + 8 = 63 \cdot 4^{3n} + 43n + 8 = 9(7 \cdot 4^{3n} + p).
\]

Therefore, \( 9|(4^{3(n+1)} + 8) \). By Mathematical Induction \( 9|(4^{3n} + 8) \) holds for any integer \( n \geq 0 \). \( \square \)

3. (Chapter 10: Question 18) Suppose \( A_1, A_2, \ldots, A_n \) are sets in some universal set \( U \), and \( n \geq 2 \). Prove that
\[
\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = A_1 \cap A_2 \cap \cdots \cap A_n.
\]

**Proof.** We will prove this by using induction on \( n \).

**Base step:** \( n = 2 \). We know whenever we have two sets \( A_1 \) and \( A_2 \) in the universal set \( U \) by DeMorgan’s law \( \overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2} \).

**Inductive step:** Let \( n \geq 2 \). We will assume that give any \( n \) subsets \( B_1, B_2, \ldots, B_n \) of \( U \) that \( B_1 \cup B_2 \cup \cdots \cup B_n = B_1 \cap B_2 \cap \cdots \cap B_n \). Our goal is to show that given any \( n + 1 \) subsets \( A_1, A_2, \ldots, A_{n+1} \) of \( U \) that \( \overline{A_1 \cup A_2 \cup \cdots \cup A_{n+1}} = A_1 \cap A_2 \cap \cdots \cap A_{n+1} \). So let \( A_1, A_2, \ldots, A_{n+1} \) be any \( n + 1 \) sets which are subsets of the universal set \( U \). Note that then \( A_1, A_2, \ldots, A_n \) is a collection of \( n \) subsets of \( U \) so
we can apply our inductive assumption to this collection of sets. Using DeMorgan’s law (page 145 in textbook, Exercise 11) and our inductive assumption we have that

\[ A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} = (A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1} \]
\[ = (A_1 \cup A_2 \cup \cdots \cup A_n) \cap A_{n+1} \]
\[ = A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1} \]

which proves our statement by induction.

\[ \square \]

4. (Chapter 10: Question 22) If \( n \in \mathbb{N} \), then

\[ \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{1}{4} \right) \left( 1 - \frac{1}{8} \right) \left( 1 - \frac{1}{16} \right) \cdots \left( 1 - \frac{1}{2^n} \right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}. \]

Proof. We will prove this by using induction on \( n \).

Base step: When \( n = 1 \) the left hand side is \( (1 - \frac{1}{2}) = \frac{1}{2} \). When \( n = 1 \) the right hand side is \( \frac{1}{4} + \frac{1}{2^{1+1}} = \frac{1}{2} \) and since \( \frac{1}{2} \geq \frac{1}{2} \) the inequality is true for \( n = 1 \).

Inductive step: Let \( n \geq 1 \). We will assume that \( (1 - \frac{1}{2}) (1 - \frac{1}{4}) (1 - \frac{1}{8}) (1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \geq \frac{1}{4} + \frac{1}{2^{n+1}} \). Our goal is to show that \( (1 - \frac{1}{2}) (1 - \frac{1}{4}) (1 - \frac{1}{8}) (1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \geq \frac{1}{4} + \frac{1}{2^{(n+1)+1}} \). Using our inductive assumption we have

\[ \left[ \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{1}{4} \right) \left( 1 - \frac{1}{8} \right) \left( 1 - \frac{1}{16} \right) \cdots \left( 1 - \frac{1}{2^n} \right) \right] \left( 1 - \frac{1}{2^{n+1}} \right) \geq \left[ \frac{1}{4} + \frac{1}{2^{n+1}} \right] \left( 1 - \frac{1}{2^{n+1}} \right) \]
\[ = \frac{1}{4} + \frac{1}{2^{n+1}} - \frac{1}{4} \frac{1}{2^{n+1}} \frac{1}{2^{n+1}} \frac{1}{2^{n+1}} \]
\[ = \frac{1}{4} + \frac{1}{2^{n+1}} \left( 1 - \frac{1}{4} - \frac{1}{2^{n+1}} \right) \]
\[ = \frac{1}{4} + \frac{1}{2^{n+1}} \left( \frac{3}{4} - \frac{1}{2^{n+1}} \right) \]

Because \( n \geq 1 \) and \( 2^{1+1} \leq 2^{n+1} \) we know \( \frac{1}{2^{n+1}} \leq \frac{1}{2^{1+1}} = \frac{1}{4} \) so

\[ \frac{1}{4} + \frac{1}{2^{n+1}} \left( \frac{3}{4} - \frac{1}{2^{n+1}} \right) \geq \frac{1}{4} + \frac{1}{2^{n+1}} \left( \frac{3}{4} - \frac{1}{4} \right) \]
\[ = \frac{1}{4} + \frac{1}{2^{n+1}} \left( \frac{1}{2} \right) \]
\[ = \frac{1}{4} + \frac{1}{2^{(n+1)+1}} \]

Thus we have shown \( (1 - \frac{1}{2}) (1 - \frac{1}{4}) (1 - \frac{1}{8}) (1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^{n+1}}) \geq \frac{1}{4} + \frac{1}{2^{(n+1)+1}} \). Hence by induction \( (1 - \frac{1}{2}) (1 - \frac{1}{4}) (1 - \frac{1}{8}) (1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \geq \frac{1}{4} + \frac{1}{2^{n+1}} \) for all \( n \in \mathbb{N} \).

\[ \square \]

5. The Fibonacci numbers are defined to be \( F_1 = 1, F_2 = 1 \), and \( F_n = F_{n-1} + F_{n-2} \) for \( n > 2 \). Show that for all \( k \in \mathbb{N} \), \( F_{4k} \) is a multiple of 3.

Proof. We will prove this by inducting on \( k \).

Base step: When \( k = 1 \) we have \( m = 4k = 4 \) and \( F_m = F_4 = F_3 + F_2 = (F_2 + F_1) + F_2 = (1+1) + 1 = 3 \) which is a multiple of 3.
Inductive step: Let \( k \geq 1 \) we will assume that \( F_{4k} \) is a multiple of 3 and show that \( F_{4(k+1)} \) is also a multiple of 3. Since \( F_{4k} \) is a multiple of 3 we know \( F_{4k} = 3x \) for some integer \( x \). Using the Fibonacci recurrence we have

\[
F_{4(k+1)} = F_{4k+4} \\
= F_{4k+3} + F_{4k+2} \\
= (F_{4k+2} + F_{4k+1}) + (F_{4k+1} + F_{4k}) \\
= ((F_{4k+1} + F_{4k}) + F_{4k+1}) + (F_{4k+1} + F_{4k}) \\
= 3F_{4k+1} + 2F_{4k} \\
= 3F_{4k+1} + 2(3x) \\
= 3(F_{4k+1} + 2x).
\]

Hence \( F_{4(k+1)} \) is also a multiple of 3. Therefore, by induction we can conclude that \( F_{4k} \) is a multiple of 3 for all \( k \in \mathbb{N} \).

6. Use induction to prove that if \( A \) is a finite set of cardinality \( n \geq 0 \) then \( |P(A)| = 2^n \).

**Proof.** We proceed by induction on \( n \).

**Base case:** If \( n = 0 \), then \( A = \emptyset \). Then \( P(A) = \{\emptyset\} \). Thus \( |A| = 0 \) and \( |P(A)| = 1 = 2^0 \). Hence the statement is true for \( n = 0 \).

**Inductive step:** Assume that the statement is true for any set of cardinality \( k \). Let \( B \) be a set with \( |B| = k + 1 \). We write \( B \) as \( B = \{b_1, ..., b_k, b_{k+1}\} \). For any subset \( C \) of \( B \), we consider two cases:

- If \( b_{k+1} \notin C \), then \( C \subseteq \{b_1, ..., b_k\} \); it follows from the induction hypothesis that there are \( 2^k \) such subsets.
- If \( b_{k+1} \in C \), then \( C = D \cup \{b_{k+1}\} \) for some \( D \subseteq \{b_1, ..., b_k\} \); again, by the induction hypothesis there are \( 2^k \) such sets \( D \).

Since these two cases give disjoint collections of sets, there must be \( 2^k + 2^k = 2^{k+1} \) subsets of \( B \). Now by induction the statement is true for all \( n \geq 0 \).

7. Let \( f(x) = x \ln x \) and \( x > 0 \). Denote \( f^{(n)}(x) \) the \( n \)th derivative of \( f(x) \) for \( n \in \mathbb{N} \). Prove that for all integer \( n \geq 3 \) it holds

\[
f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}.
\]

**Proof.** We use induction on \( n \).

**Base case:** \( n = 3 \). Direct computation shows:

\[
f'(x) = 1 + \ln x \\
f''(x) = \frac{1}{x} \\
f^{(3)}(x) = -\frac{1}{x^2} = (-1)^3 \frac{(3-2)!}{x^{3-1}}.
\]

Thus the statement is true for \( n = 3 \).

**Inductive step:** Assume \( k \geq 3 \) and

\[
f^{(k)}(x) = (-1)^k \frac{(k-2)!}{x^{k-1}}.
\]
For $k + 1$, we differentiate the above equality
\[
f^{(k+1)}(x) = (f^{(k)}(x))' = (-1)^k(k-2)!(k-1)(-1)x^{-(k-1)-1} = (-1)^{k+1}(k-1)!x^{-k} = (-1)^{k+1} \frac{(k+1) - 2)!}{x^{(k+1)-1}}.
\]

By induction the statement is true for all integer $n \geq 3$. \hfill \Box

8. A sequence $a_1, a_2, ..., a_n, ...$ is defined by
\[a_1 = 1, a_2 = 4, \text{ and } a_n = 2a_{n-1} - a_{n-2} + 2 \text{ for all } n \geq 3.\]
Show: $a_n = n^2$ for all $n \in \mathbb{N}$.

**Proof.** We proceed by strong induction. Since $a_1 = 1 = 1^2$ and $a_2 = 4 = 2^2$, the formula holds for $n = 1, 2$. Assume $a_k = k^2$ for every integer $k$ with $2 \leq k \leq n$. As $n + 1 \geq 3$, we can use the iteration formula as follows
\[
a_{n+1} = 2a_n - a_{n-1} + 2
= 2n^2 - (n-1)^2 + 2
= 2n^2 - (n^2 - 2n + 1) + 2
= n^2 + 2n + 1
= (n+1)^2.
\]

By strong induction, $a_n = n^2$ for all $n \in \mathbb{N}$. \hfill \Box

9. Suppose you begin with a pile of $n$ stones ($n \geq 2$) and split this pile into $n$ piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have $p$ and $q$ stones in them, respectively, you compute $pq$. Show that no matter how you split the piles (eventually into $n$ piles of one stone each), the sum of the products computed at each step equals $n(n-1)/2$. (Hint: use strong induction on $n$.)

**Proof.** We proceed by strong induction.

**Base case:** When $n = 2$, the stones are split into two piles of one stone each. Thus $p = q = 1$ and $pq = 1 = (2)(2-1)/2$. So the statement is true for $n = 2$.

**Inductive step:** Assume that the statement is true for the game of any $l$ stones with $2 \leq l \leq n$. For $n + 1$ stones, suppose that we finish the first round of the splitting and have two piles now: one with $k$ stones and one with $n + 1 - k$ stones, $k > 0$.

Case 1: If $k = 1$ then the pile with $k = 1$ stone will not be changed in the later part of the game and the pile with $n + 1 - k = n$ falls in the range of the induction hypothesis, so the sum of the products from each step to the end of the game is
\[
1 \cdot n + \frac{n(n-1)}{2} = \frac{(n+1)n}{2}.
\]

Case 2: If $k = n$, we have a pile with $n$ stones and a pile with 1 stone, so we have the same formula for the sum of the products as in Case 1.

Case 3: If $2 \leq k < n$, we can apply the induction hypothesis to both piles since $2 \leq k \leq n$ and $2 \leq n + 1 - k \leq n$ (to see this: as $k < n$, $k + 1 \leq n$, so $1 \leq n - k$, then $2 \leq n - k + 1$; adding $n$ to both sides of $1 - k \leq 0$ yields $n + 1 - k \leq n$). Now, the two piles together contribute a product $k(n + 1 - k)$,
and individually the two piles contribute \( \frac{k(k - 1)}{2} \) and \( \frac{(n + 1 - k)(n - k)}{2} \), respectively, according to the induction hypothesis; therefore the total sum is

\[
\frac{k(k - 1)}{2} + \frac{(n + 1 - k)(n - k)}{2} + k(n + 1 - k)
\]

\[
= \frac{1}{2} \left( (k^2 - k) + ((n + 1)n - (n + 1)k - kn + k^2) + 2k(n + 1 - k) \right)
\]

\[
= \frac{(n + 1)n}{2}.
\]

Hence, by strong induction, the statement holds for any integer \( n \geq 2 \).