A useful criterion to determine a set is countable

A set is countable if it is either countably infinite or finite.

**Theorem 1.** Let $S$ be a nonempty set. The following three conditions are equivalent.

1. $S$ is countable.
2. There is an injection $f : S \to \mathbb{N}$.
3. There is a surjection $g : \mathbb{N} \to S$.

**Proof.** We will show that (1) implies (2) implies (3) implies (1).

(1) implies (2): Assume that $S$ is countable. If $S$ is finite, then there is a bijection $h : \{1, 2, \ldots, |S|\} \to S$, where $|S| \geq 1$ since $S \neq \emptyset$. Define $f : S \to \mathbb{N}$ by $f(s) = h^{-1}(s)$ for $s \in S$. Then $f$ is injective. If $S$ is countably infinite, then there is a bijection $H : \mathbb{N} \to S$. Then $H^{-1} : S \to \mathbb{N}$ is injective, so we take $f = H^{-1}$.

(2) implies (3): Suppose that $f : S \to \mathbb{N}$ is an injection. Then $f$ is a bijection from $S$ to $f(S)$. So $f^{-1}$ is a bijection from $f(S)$ to $S$ (note that we do not know $f$ is surjective as a function from $S$ to $\mathbb{N}$). Fix an element $s_0 \in S$. Define $g : \mathbb{N} \to S$ by

$$
g(n) = \begin{cases} 
  f^{-1}(n), & \text{if } n \in f(S) \\
  s_0, & \text{if } n \notin f(S).
\end{cases}
$$

Then $g(f(S)) = f^{-1}(f(S)) = S$ and $g(\mathbb{N} - f(S)) = \{s_0\}$. So $g$ is a surjective.

(3) implies (1): Suppose that $g : \mathbb{N} \to S$ is a surjection. Define $h : S \to \mathbb{N}$ by $h(s) =$ the smallest $n \in \mathbb{N}$ such that $g(n) = s$ (note the Well-ordering principle guarantees existence of such $n$). Then $h$ is injective: If $h(s_1) = h(s_2)$, then $g(h(s_1)) = s_1$ and $g(h(s_2)) = s_2$ imply that $s_1 = s_2$. Since $\mathbb{N}$ is countable and $h(S) \subseteq \mathbb{N}$, $h(S)$ is countable: either finite or countably infinite by Theorem 13.8 if it is infinite. Since $h$ is a bijection from $S$ to $h(S)$ (not necessarily surjective to $\mathbb{N}$), $S$ is countable. □

**Theorem 2.** Let $A$ and $B$ are countable sets. Then $A \cup B$ is countable.

**Proof.** If at least one of $A$ and $B$ is $\emptyset$, say $A = \emptyset$, then $A \cup B = B$ is countable. Now assume neither $A$ nor $B$ is $\emptyset$. Since $A$ is countable, there is a surjection $f : \mathbb{N} \to A$ (if $A$ is countably infinite there is a bijection (so surjective in particular) from $\mathbb{N}$ to $A$, if $A \neq \emptyset$ is finite then there is $n \in \mathbb{N}$ such that there is a bijection from $\{1, \ldots, n\}$ to $A$ and this can be composted with the surjection from $\mathbb{N}$ to $\{1, \ldots, n\}$ which is the identity function on $\{1, \ldots, n\}$ and takes all other elements in $\mathbb{N}$ to 1 to get a surjection from $\mathbb{N}$ to $A$). Similarly there is a surjection $g : \mathbb{N} \to B$. Now define $h : \mathbb{N} \to A \cup B$ by $h(n) = f(\frac{n+1}{2})$ if $n$ is odd and $h(n) = g(\frac{n}{2})$ is $n$ is even. By the previous theorem, we will be done if $h$ is surjective. For any $x \in A \cup B$, $x \in A$ or $x \in B$. If $x \in A$, then $x = f(m)$ for some $m \in \mathbb{N}$ as $f$ is surjective, so $h(2m - 1) = f(\frac{(2m-1)+1}{2}) = f(m) = x$. If $x \in B$, then $x = g(k)$ for some $k \in \mathbb{N}$ as $g$ is surjective. So $h(2k) = g(k) = x$. We now see that $h$ is indeed surjective. □