*This midterm has **5 questions** on **6 pages** Duration: 50 minutes*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First): _____

Student Number: _____

Signature: _____

Question:	1	2	3	4	5	Total
Points:	15	8	8	11	8	50
Score:						

15 marks 1. (a) Define precisely what it means for a nonempty set S of the real numbers to be well ordered.

Every nonempty subset of S has a least element

Is the interval $[0, 2]$ well ordered? Explain.

No, $(0,1) \subset [0,2]$, there is no least element in $(0,1)$.

(b) Show that $\sqrt{2}$ is irrational.

Assume $\sqrt{2}$ is in $\mathbb{Q}$, then $\exists\, m, n \in \mathbb{Z}$, $n \neq 0$ m, n have no common divisor other than ± 1, s.t. $\sqrt{2} = \frac{m}{n}$. then $2n^2 = m^2$. So m^2 is even so m is even (a result proved in class) so $m = 2k$ for some $k \in \mathbb{Z}$ Thus, $2n^2 = 4k^2$. $\Rightarrow$ $n^2 = 2k^2$ is even $\therefore$ n is even $\therefore$ 2 is a common divisor of m, n. Contradicts the way m, n are chosen. $\therefore$ $\sqrt{2}$ is irrational.

(c) Determine whether the following statement is True or False. Justify your answer.

"Every equivalence relation R on a nonempty set A is a function from A to A."

False. Counter-example.

$A = \{1, 2\}$.

$R = \{(1,1), (2,2), (1,2), (2,1)\}$

R is an equiv. rel. on A

R is not a function from A to A

as $(1,1)$, $(1,2)$ are in R

(or say $(2,2)$, $(2,1)$ are in R)

8 marks 2. Let R be a relation on $\mathbb{N}$ which is defined by $m\,R\,n$ if m^2-n^2 is even. Is R an equivalence relation on $\mathbb{N}$? If the answer is yes, find all distinct equivalence classes of R. Justify your answer.

reflexive: $\forall\, m \in \mathbb{N}$, m^2-m^2 is 0 hence even
$\therefore\ m\,R\,m$

symm: If $m\,R\,n$, then m^2-n^2 is even
$\therefore\ n^2-m^2$ is even
$\therefore\ n\,R\,m$

transitiv: If $m\,R\,n$, $n\,R\,k$, then
m^2-n^2 is even ①
n^2-k^2 is even ②
$\therefore\ m^2-k^2$ is even by adding ① and ②
$\therefore\ m\,R\,k$
$\therefore\ R$ is an equiv. rel. on $\mathbb{N}$.

$[1] = \{n \in \mathbb{N} \mid n\,R\,1\} = \{n \in \mathbb{N} \mid n^2-1 \text{ even}\}$
$= \{n \in \mathbb{N} \mid n^2 \text{ odd}\} = \{n \in \mathbb{N} \mid n \text{ odd}\}$
$[2] = \{n \in \mathbb{N} \mid n\,R\,2\} = \{n \in \mathbb{N} \mid n^2-2 \text{ even}\}$
$= \{n \in \mathbb{N} \mid n^2 \text{ even}\} = \{n \in \mathbb{N} \mid n \text{ even}\}$

Now, $\{[1],[2]\}$ is a partition of $\mathbb{N}$
$\therefore$ There're only two distinct equiv. classes:
$[1]$, $[2]$.

8 marks 3. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined by $f(n) = n^2 + n$ for $n \in \mathbb{N}$.

(a) Is f injective?

Yes

If $f(m) = f(n)$, then $m^2 + m = n^2 + n$

$\therefore \quad m^2 - n^2 + m - n = 0$

$\therefore \quad (m-n)(m+n) + (m-n) = 0$

$\therefore \quad (m-n)(m+n+1) = 0$

Since $m, n \in \mathbb{N}$, $m+n+1 \neq 0$

$\therefore \quad m = n$

$\therefore$ f is injective.

(b) Is f surjective?

No

Take $1 \in \mathbb{N}$. Since for any $n \in \mathbb{N}$

$n^2 + n = n(n+1)$ is even (if n even, $n(n+1)$ even; if n odd, $n+1$ is even so $n(n+1)$ is even)

$\therefore \quad n(n+1) \neq 1$

i.e. <u>no</u> $n \in \mathbb{N}$, s.t. $n^2 + n = 1$.

11 marks 4. (a) State carefully the Principle of Mathematical Induction.

$\forall n \in \mathbb{N}$, $P(n)$ is a statement. If
1) $P(1)$ is true 2) $\forall k \in \mathbb{N}$, if $P(k)$ true then $P(k+1)$ true,
then $P(n)$ is true for all $n \in \mathbb{N}$.

(b) Use Mathematical Induction to prove: $1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$ for all $n \in \mathbb{N}$.

$n = 1$: LHS $= 1 \cdot 3 = 3$
RHS $= \frac{1}{6} 1 (1+1)(2 \cdot 1 + 7) = 3$. $P(1)$ true.

Assume $1 \cdot 3 + \cdots + k(k+2) = \frac{1}{6} k(k+1)(2k+7)$, for $k \in \mathbb{N}$.

$k+1$:

$$\begin{aligned}
\text{LHS} &= 1 \cdot 3 + k \cdot (k+2) + (k+1)(k+3) \\
&= \frac{1}{6} k(k+1)(2k+7) + (k+1)(k+3) \\
&= \frac{1}{6}(k+1)\left[k(2k+7) + 6(k+3)\right] \\
&= \frac{1}{6}(k+1)\left[2k^2 + 13k + 18\right] \\
&= \frac{1}{6}(k+1)(k+2)(2k+9) \\
&= \frac{1}{6}(k+1)(k+2)\left[2(k+1) + 7\right] \\
&= \text{RHS}.
\end{aligned}$$

By Principle of Math. Induction, we're done.

8 marks 5. Let $f : A \to B$ be a function and let $C \subseteq A$ and $C \neq \emptyset$.

(a) Prove: if f is injective, then $f(A - C) \subseteq B - f(C)$.

$\forall y \in f(A-C)$. $\exists x \in A-C$ s.t. $y = f(x)$

since $x \notin C$, $f(x) \notin f(C)$ because if

$f(x) \in f(C)$, then $f(x) = f(c)$ for some $c \in C$

Since f injective, $x = c$, $c \in C \Rightarrow x \in C$

contradicts $x \notin C$.

$\therefore$ $y \notin f(C)$. $y \in B$ as $f(A-C) \subseteq B$.

$\therefore$ $f(A-C) \subseteq B - f(C)$

(If $C = A$, then $f(A-C) = \phi \subseteq B - f(C)$)

(b) Prove: if f is surjective, then $B - f(C) \subseteq f(A - C)$.

$\forall y \in B - f(C)$, since f is surjective,

$\exists x \in A$, s.t. $y = f(x)$

If $x \in C$, then $y \in f(C)$ contradicts $y \in B - f(C)$

So $x \notin C$, so $x \in A - C$ $\therefore$ $y = f(x) \in f(A-C)$.

(If $C = A$, $f(C) = B$ since f is onto

$\therefore$ $B - f(C) = \phi \subseteq \underbrace{f(A-C)}_{\phi}$)