*This midterm has **4 questions** on **5 pages***

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (including all middle names): _____

Student-No: _____

Signature: _____

1. (a) Write the negation of the following statement:

"There is some $(a, b) \in \mathbb{N} \times \mathbb{N}$ so that $a \leq b$ and $a^2 \leq b^2$."

Solution: $\forall (a, b) \in \mathbb{N} \times \mathbb{N}, (a > b)$ or $(a^2 > b^2)$.

(b) Write the negation of the following statement:

"The integer n is even if and only if $n^2 + 1$ is even."

Solution: "The integer n is even if and only if $n^2 + 1$ is odd."

(c) Write the converse and contrapositive of the following statement:

"If n is even, then $n + 1$ is odd."

Solution: Converse — "If $n + 1$ is odd, then n is even."
Contrapositive — "If $n + 1$ is even, then n is odd."

(d) Give a precise definition of a *set partition*.

Solution:
Definition. Let A be a set. A partition of A is a collection S of subsets of A such that

- every $X \in S$ is non-empty
- for every $X, Y \in S$, either $X = Y$ or $X \cap Y = \varnothing$
- $\bigcup_{X \in S} X = A$.

2. (a) Let $S = \{x \in \mathbb{R} \text{ s.t. } x > 0\}$.
Determine whether the following six statements are true or false — explain your answers ("true" or "false" is not sufficient).

(i) $\forall x \in S, \forall y \in S, xy = 3$.

(ii) $\forall x \in S, \exists y \in S$ s.t. $xy = 3$.

(iii) $\exists x \in S$ s.t. $\forall y \in S, xy = 3$.

(iv) $\exists x \in S$ s.t. $\exists y \in S$ s.t. $xy = 3$.

(v) $\exists x \in S$ s.t. $\forall y \in S, \forall z \in S$, if $z > y$, then $z \geq x + y$.

(vi) $\forall x \in S, \exists y \in S$ s.t. $\forall z \in S$, if $z > y$, then $z \geq x + y$.

Solution:

(i) False — the negation is $\exists x, \exists y$ s.t. $xy \neq 3$. Pick $x = 1, y = 1$ then $xy = 1 \neq 3$. Since the negation is true, the original must be false.

(ii) True — for any x, pick $y = 3/x$. Then no matter what choice of x, $xy = 3$.

(iii) False — the negation is $\forall x, \exists y$ s.t. $xy \neq 3$. For any given x, pick $y = 1/x$, then $xy = 1 \neq 3$. Since the negation is true, the original must be false.

(iv) True — take $x = 1, y = 3$.

(v) False — the negation is $\forall x \in S$, $\exists y \in S, \exists z \in S$, s.t. $z > y$ and $z < x + y$. Take $z = y + \frac{x}{2}$. So $z > y$ since $x/2 > 0$, and $z < x + y$. The negation is true, the original is false.

(vi) False — choose $x = 2$. For any choice of $y \in S$, if $z = 1 + y$, then $z > y$, but $1 + y \not\geq 2 + y$.

(b) Give an example of two sets, A, B, such that $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

Solution: $A = \{1\}, B = \{2\}$. Then $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{1\}, \{2\}\}$, while $\mathcal{P}(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

3. (a) Let $n \in \mathbb{Z}$. Prove that $7n + 1$ and $3n - 6$ have opposite parity.
(That is, they cannot both be odd and they cannot both be even).

Solution: The number n is either even or odd and so we consider the two cases.

- Assume n is even, so $n = 2k$ for some $k \in \mathbb{Z}$. Then $7n + 1 = 14k + 1 = 2(7k) + 1$ and thus odd. On the other hand $3n - 6 = 6k - 6 = 2(3k - 3)$ and thus even. Hence $7n + 1$ and $3n - 6$ have opposite parity.
- Now assume that n is odd, so $n = 2l + 1$ for some $l \in \mathbb{Z}$. Then $7n + 1 = 14l + 8 = 2(7l + 4)$ and thus even. On the other hand $3n - 6 = 6l - 3 = 2(3l - 2) + 1$ and thus odd. Hence $7n + 1$ and $3n - 6$ have opposite parity.

(b) Prove that $a^3 + 2a \equiv 0 \pmod 3$.

Solution:

Proof. We proceed by direct proof. We consider the three possible cases:

- If $a \equiv 0 \pmod 3$, then $a^3 + 2a \equiv 0^3 + (2)(0) \pmod 3$.
- If $a \equiv 1 \pmod 3$, then $a^3 + 2a \equiv 1^3 + (2)(1) \pmod 3$. Then since $3 \equiv 0 \pmod 3$, $a^3 + 2a \equiv 0 \pmod 3$.
- If $a \equiv 2 \pmod 3$, then $a^3 + 2a \equiv 2^3 + (2)(2) \pmod 3$. Then since $12 \equiv 0 \pmod 3$, again $a^3 + 2a \equiv 0 \pmod 3$.

□

4. Let A, B, C be sets.

   (a) Prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Solution:

Proof.

- Let $x \in \overline{A \cap B}$, and so $x \notin A \cap B$.
- This implies (by de Morgan's laws) that $x \notin A$ or $x \notin B$.
- Hence $x \in \overline{A}$ or $x \in \overline{B}$.
- And so $x \in \overline{A} \cup \overline{B}$.

□

   (b) Prove that $(A - B) \cup (A - C) \subseteq \big(A - (B \cap C)\big)$

Solution:

Proof. Assume $x \in (A - B) \cup (A - C)$.

- Hence $x \in (A - B)$ or $x \in (A - C)$.
  - If $x \in (A - B)$ then $x \in A$ and $x \notin B$. Since $x \notin B$, $x \notin B \cap C$.
  - If $x \in (A - C)$ then $x \in A$ and $x \notin C$. Since $x \notin C$, $x \notin B \cap C$.
- In either case, $x \in A$ and $x \notin B \cap C$, and thus $x \in A - (B \cap C)$.
- Hence $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.

□

   (c) Prove that $(A \cap B) \cup (A \cap C) \subseteq \big(A \cap (B \cup C)\big)$.

Solution:

Proof. Let $x \in (A \cap B) \cup (A \cap C)$.

- So $x \in A \cap B$ or $x \in A \cap B$.
  - If $x \in A \cap B$, then $x \in A$ and $x \in B$. Thus $x \in B \cup C$ and so $x \in A \cap (B \cup C)$.
  - Similarly, if $x \in A \cap C$, then $x \in A$ and $x \in C$. Thus $x \in B \cup C$ and so $x \in A \cap (B \cup C)$.
- In either case $x \in A \cap (B \cup C)$.
- Hence $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

□