If you are using the 2nd edition, be careful — question numbers may not agree.

- 8.6, 8.12, 8.28, 8.32, 8.38, 8.40, 8.42, 8.46, 8.50
- EQ1 Let A be the set $\{1, 2, 3\}$. Answer the following:
  (a) Consider the relation $\{(1, 1), (2, 2), (3, 1), (3, 3)\}$ on A. Determine whether or not it is an equivalence relation. If it is not, state which properties it is lacking. If it is, describe the partition it determines by listing the subsets of the partition.
  (b) Repeat (a) for the relation $\{(1, 1), (2, 3), (2, 2), (3, 2), (3, 3)\}$ on A.
- EQ2 Let $A = \mathbb{N} \times \mathbb{N}$, and define a relations R on A by $(a, b)R(c, d)$ if and only if $ab = cd$.
  (a) Show that R is an equivalence relation on A.
  (b) List the elements in the equivalence class $[(9, 2)]$.
  (c) Find an equivalence class with exactly two elements.
  (d) Find an equivalence class with exactly three elements.

8.6 $cR^{-1}d$ if and only if dRc. By definition dRc when $d/c \in \mathbb{N}$. Hence $cR^{-1}d$ iff $c|d$.

8.12 We check whether R is reflexive, symmetric and transitive.

- Reflexive — it is not reflexive because $(b, b) \notin R$.
- Symmetric — it is not symmetric because $(a, b) \in R$ but $(b, a) \notin R$.
- Transitive — it is transitive. We need to verify that for every choice of x, y, z so that $(x, y) \in R$ and $(y, z) \in R$, we also have $(x, z) \in R$. The only such choice for (x, y) is $(x, y) = (a, a)$. Then the only choices for (y, z) are $(a, a), (a, b)$ and (a, c). In each case $(x, z) = (y, z) \in R$.

8.28 Part(a): We must prove that the relation is reflexive, symmetric and transitive.

- Reflexive — Let $a \in \mathbb{Z}$, then $a + a = 2a$ which is even. Hence aRa.
- Symmetric — Let $a, b \in \mathbb{Z}$ and assume that aRb. Hence $a + b$ is even. But $a + b = b + a$, so $b + a$ is also even. Hence bRa.
- Transitive — Let $a, b, c \in \mathbb{Z}$ and assume that aRb and bRc. Hence $a + b = 2k, b + c = 2l$ for some $k, l \in \mathbb{Z}$. Now

$$\begin{aligned} a + c &= (a + b) + (b + c) - 2b \\ &= 2k + 2l - 2b = 2(k + l - b) \end{aligned}$$

  Since $k + l - b \in \mathbb{Z}$, it follows that $a + c$ is even and so aRc as required.

The distinct equivalence classes are

$$\begin{aligned} [0] &= \{x \in \mathbb{Z} \mid xR0\} \\ &= \{x \in \mathbb{Z} \mid x + 0 \text{ is even}\} \\ &= \{x \in \mathbb{Z} \mid x \text{ is even}\} \\ [1] &= \{x \in \mathbb{Z} \mid xR1\} \\ &= \{x \in \mathbb{Z} \mid x + 1 \text{ is even}\} \\ &= \{x \in \mathbb{Z} \mid x \text{ is odd}\} \end{aligned}$$

Part(b):

- The relation is not reflexive — $0 \not R\, 0$ since $0+0$ is not odd.
- The relation is symmetric — if $a+b$ is odd then $b+a=a+b$ is odd.
- The relation is not transitive — $0R1$ and $1R2$ but $0 \not R\, 2$, since $0+2=2$ which is not odd.

8.32 A relation R is defined on the set $A = \{a+b\sqrt{2} \,:\, a,b \in \mathbf{Q}, a+b\sqrt{2} \neq 0\}$ by $x\ R\ y$ if $x/y \in \mathbf{Q}$. Show that R is an equivalence relation and determine the distinct equivalence classes.

Solution:

- **reflexive:** Given $x \in A$, we clearly have $x/x = 1 \in \mathbf{Q}$; so $x\ R\ x$.
- **symmetric:** Assume $x, y \in A$ are given and $x\ R\ y$. By definition of the relation $x/y \in \mathbf{Q}$. Because $x \neq 0$, this implies that x/y is a nonzero rational number and therefore the inverse $\dfrac{1}{x/y} = y/x$ is also a rational number. Then again by the definition of R, this implies $y\ R\ x$.
- **transitive:** Assume $x, y, z \in A$ are given with $x\ R\ y$ and $y\ R\ z$. By the definition of R, equivalently x/y and y/z are rational numbers. Product of rational numbers is rational, so $(x/y)\cdot(y/z) = x/z$ is also rational. By the definition of R, $x\ R\ z$.

We provide distinct equivalence classes by providing a unique representative from each of them. Consider the subset

$$B = \{r+\sqrt{2} \,:\, r \in \mathbf{Q}\} \cup \{1\}$$

of A. We claim every element of A is equivalent to exactly one of the elements of A and therefore elements of B represent all distinct equivalence classes of R. Given $x = a+b\sqrt{2} \in A$ for $a, b \in \mathbf{Q}$. If $b=0$ then $x=a$ and therefore $x/1 = a \in \mathbf{Q}$; hence $xR1$. If $b \neq 0$, then $r = a/b \in \mathbf{Q}$ and we can see that

$$\frac{x}{r+\sqrt{2}} = \frac{a+b\sqrt{2}}{(a/b)+\sqrt{2}} = b \in \mathbf{Q}.$$

Therefore $x\ R\ (r+\sqrt{2})$.

It only remains to show that distinct elements of B are not equivalent. Suppose $r+\sqrt{2}$ and $s+\sqrt{2}$ are distinct elements of B. Let

$$K = \frac{r+\sqrt{2}}{s+\sqrt{2}} = \left(\frac{r+\sqrt{2}}{s+\sqrt{2}}\right)\left(\frac{s-\sqrt{2}}{s-\sqrt{2}}\right) = \frac{rs-2+(s-r)\sqrt{2}}{s^2-2}.$$

From this we conclude

$$\sqrt{2} = \frac{K(s^2-2)-(rs-2)}{s-r}$$

is rational which we know is not the case. Hence $r+\sqrt{2}$ and $s+\sqrt{2}$ are not equivalent. Also we can see that for every element $r+\sqrt{2}$ of B, $(r+\sqrt{2})/1 = r+\sqrt{2}$ is also irrational and therefore $r+\sqrt{2}$ is not equivalent to 1. Hence every two distinct elements of B are not equivalent and represent different equivalence classes.

8.38 Let R be a relation defined on the set $\mathbf{N}$ by $a\ R\ b$ if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: R is an equivalence relation.

Solution: It is not an equivalence relation and is not transitive. It is easy to see that $12\ R\ 6$ and $6\ R\ 9$ but neither $12|(2\times 9)$ nor $9|(2\times 12)$.

8.40 A relation R is defined on $\mathbf{Z}$ by $x\ R\ y$ if $3x-7y$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

Solution:

- **reflexive:** Given $x \in \mathbf{Z}$, $3x-7x = -4x = 2(-2x)$ is even and therefore $x\ R\ x$.
- **symmetric:** Suppose $x, y \in \mathbf{Z}$ are given and $x\ R\ y$. By definition of R, $3x-7y$ is even. Then
$$3y-7x = -(3x-7y) - 2(2x+2y)$$
is the sum of two even numbers and therefore is even. Hence $y\ R\ x$.
- **transitive:** Finally suppose $x, y, z \in \mathbf{Z}$ are given with $x\ R\ y$ and $y\ R\ z$. Equivalently $3x-7y$ and $3y-7z$ are both even. Then
$$3x-7z = (3x-7y) + (3y-7z) + 4y$$
is a sum of three even numbers and is therefore even. Hence $x\ R\ z$.

We claim there are exactly two equivalence classes. All even numbers are in one equivalence class and the odd equivalence classes are in the other. If $x = 2k$ is even then we claim that $x\ R\ 0$. This is because
$$3x - 7\times 0 = 6k = 2(3k)$$
is even. On the other hand if $x = 2k+1$ is odd, we claim that $x\ R\ 1$. This is because
$$3x - 7\times 1 = 3(2k+1) - 7 = 6k - 4 = 2(3k-2)$$
is even. Therefore every even number is equivalent to 0 and every odd number is equivalent to 1. On the other hand 0 and 1 are not equivalent since
$$3\times 0 - 7\times 1 = -7$$
is odd.

8.42 Prove or disprove: *The union of two equivalence relations on a nonempty set is an equivalence relation.*

Solution: This is false. On the set $\{1, 2, 3\}$ consider the equivalence relations

$$R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\} \qquad R_2 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$$

Then $R_1 \cup R_2$ contains $(1, 2)$ and $(2, 3)$ but does not contain $(1, 3)$, so it fails to be transitive and therefore is not an equivalence relation.

8.46. Let R be the relation defined by $\mathbf{Z}$ by $a\ R\ b$ if $a + b \equiv 0 \pmod 3$. Show that R is not an equivalence relation.

Solution: R is not an equivalence relation. It is not reflexive, since, for example, $1 + 1 = 2$, which is not congruent to $0 \pmod 3$, and so we do *not* have $1\ R\ 1$.

8.50. A relation R is defined on $\mathbf{Z}$ by $a\ R\ b$ if $2a + 2b \equiv 0 \pmod 4$. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

Solution: R is an equivalence relation. It is:

- **reflexive**: For any $a \in \mathbf{N}$, we have $2a + 2a = 4a$. Since $4a$ is divisible by 4, it is congruent to $0 \pmod 4$. It follows that $a\ R\ a$.
- **symmetric**: For any $a, b \in \mathbf{N}$, we have $2a + 2b = 2b + 2a$. It follows that if $a\ R\ b$ then $b\ R\ a$.
- **transitive**: For any $a, b, c \in \mathbf{N}$, we have $(2a+2b)+(2b+2c) = (2a+2c)+4b \equiv 2a + 2c \pmod 4$, since $4b \equiv 0 \pmod 4$. Then if $2a + 2b \equiv 0 \pmod 4$ and $2b + 2c \equiv 0 \pmod 4$, it follows that $2a + 2c \equiv 0 \pmod 4$.

Note that for any integers a, b, that $2a + 2b = 2(a + b)$, which is divisible by 4 if and only if $a + b$ is even, which happens if and only if either a and b are both even, or a and b are both odd. The equivalence classes of R are $[0]$, consisting of all even integers, and $[1]$, consisting of all odd integers.

EQ1 Let A be the set $\{1, 2, 3\}$. Answer the following:

(a) Consider the relation $\{(1, 1), (2, 2), (3, 1), (3, 3)\}$ on A. Determine whether or not it is an equivalence relation. If it is not, state which properties it is lacking. If it is, describe the partition it determines by listing the subsets of the partition.

(b) Repeat (a) for the relation $\{(1, 1), (2, 3), (2, 2), (3, 2), (3, 3)\}$ on A.

Solution:

(a) The relation $\{(1,1),(2,2),(3,1),(3,3)\}$ on A in *not* an equivalence relation. It is not symmtric, since $(3,1)$ is in the set, but $(1,3)$ is not.

(b) The relation $\{(1,1),(2,3),(2,2),(3,2),(3,3)\}$ on A is an equivalence relation. It is:

- **reflexive**: $(1,1)$, $(2,2)$, and $(3,3)$ are all part of the relation.
- **symmetric**: 2 and 3 are the only numbers that are paired with another number besides themselves, and we see that $(2,3)$ and $(3,2)$ are both part of the relation.
- **transitive**: 1 is only related to itself. 2 is related to 3, and 3 is related to itself and 2, and both $(2,3)$ and $(2,2)$ are part of the relation. Likewise, 3 is related to 2, and 2 is related to itself and 3, and both $(3,2)$ and $(3,3)$ are in the relation.

The partition given by the equivalence relation consists of the sets

$$\{1\} \quad \text{and} \quad \{2,3\}$$

EQ2 Let $A = \mathbb{N} \times \mathbb{N}$, and define a relations R on A by $(a,b)R(c,d)$ if and only if $ab = cd$.

(a) Show that R is an equivalence relation on A.

(b) List the elements in the equivalence class $[(9,2)]$.

(c) Find an equivalence class with exactly two elements.

(d) Find an equivalence with exactly three elements.

Solution:

(a) R is an equivalence relation because it is:

- **reflexive**: For every $(a,b) \in A$, we have $ab = ab$, so that $(a,b)R(a,b)$.
- **symmetric**: If $a,b,c,d \in \mathbf{N}$ are such that $ab = cd$, then also $cd = ab$, since equality is symmetric. Thus $(a,b)R(c,d)$ implies $(c,d)R(a,b)$.
- **transitive**: If $a,b,c,d,m,n \in \mathbf{N}$ are such that $ab = cd$ and $cd = mn$, then $ab = mn$, since equality is transitive. Thus whenever $(a,b)R(c,d)$ and $(c,d)R(m,n)$, we have $(a,b)R(m,n)$.

(b) $[(9,2)] = \{(1,18),(18,1),(9,2),(2,9),(3,6),(6,3)\}$.

(c) $[(1,2)] = \{(1,2),(2,1)\}$ has exactly two elements.

(d) $[(1,4)] = \{(1,4),(4,1),(2,2)\}$ has exactly three elements.