\[ k'(x) = \frac{1 - 2x^2}{(x^2 + 1)^{5/2}} \Rightarrow \text{max curvature at } x = \frac{1}{\sqrt{2}}. \]

Normal and binormal vectors

(orthogonal unit vectors along each point on the curve)

all vectors that lie in the normal plane are \( \perp \) to the unit tangent vector \( \vec{T} \) at \( \vec{r} = \vec{r}_0 \).

ie, \[ \left[ (x, y, z) - \vec{r}_0 \right] \cdot \vec{T} = 0 \]

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plane is spanned by two vectors; choose them so that these two vectors and \( \vec{T} \) are mutually orthogonal; one vector, called the principal normal vector is

\[
\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}, \quad \vec{T}'(t) = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}
\]

\( |\vec{T}'| \neq 0 \)

plane curve

\( \vec{N}(t) \) points toward the "center" or concave side of the curve (the direction toward which \( \vec{T} \) is changing)
Recall $|\vec{T}| = 1 = \text{constant}$

\[ \frac{d\vec{T}}{dt} \cdot \vec{T} = 0 = 2\vec{T} \cdot \vec{T}' \]

So $\vec{T} \perp \vec{N}$.

Choose other vector to be $\perp$ to both $\vec{N}$ and $\vec{T}$ so 

$\vec{B} = \vec{T} \times \vec{N}$

$\uparrow$ Binormal vector

$|\vec{B}| = 1$, $\vec{B}$, $\vec{T}$, $\vec{N}$ mutually $\perp$.

All unit vectors.

All vectors lying in the osculating plane are $\perp$ to $\vec{B}$;

\[ [(x, y, z) - \vec{r}_0] \cdot \vec{B} = 0. \]

E.g. for a plane curve, this is just the plane that contains the curve
\( \mathbf{B} = (0, 0, 1) \)

\( \mathbf{B} \) is constant for a planar curve.

The osculating circle lies in the osculating plane, has the same tangent at \( \mathbf{r}_0 \), and has same curvature. The center of the circle located at

\[ \mathbf{x} = \mathbf{r}_0 + \frac{1}{k} \mathbf{N} \]
Ex find equ of normal and osculating planes of \( \mathbf{r}(t) = (t, t^2, t^3) \) at \( (1, 1, 1) \).

\[ \mathbf{T}_t = 1. \]

Need to calculate \( \mathbf{T} \) and \( \mathbf{B} \)

\[ \begin{align*}
\mathbf{T} &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{(1, 2t, 3t^2)}{\sqrt{1 + 4t^2 + 9t^4}} \\
\mathbf{T}(1) &= \frac{(1, 2, 3)}{\sqrt{14}}
\end{align*} \]

Eqn of normal plane is

\[ [\mathbf{x}, y, z] - (1, 1, 1) \cdot \mathbf{T} = 0 \]

Eqn of osculating plane is

\[ [\mathbf{x}, y, z] - (1, 1, 1) \cdot (1, 2, 3) = 0 \]
\[ x + 2y + 3z - 6 = 0 \]

osc. plane \quad (\text{need } \vec{B} = \vec{T} \times \vec{N}).

\[ \vec{N} = \frac{\vec{T}'}{1 + \vec{T}' \cdot \vec{T}'} \]

\[ \frac{d\vec{T}'}{dt} = \frac{d}{dt} \left( \frac{1, 2t, 3t^2}{\sqrt{1 + 4t^2 + 9t^4}} \right) \]

\[ \vec{T}'(1) = \frac{-2}{(14)^{3/2}} \left( 11, 8, -9 \right) \]

so \( \vec{N} \) points in direction \( (11, 8, -9) \)
the \( \vec{B} \) points in direction \( (1, 2, 3) \times (11, 8, -9) = (42, -42, 14) \)

\[ \text{direction of } \vec{T} \quad \text{direction of } \vec{N} \]

so eqn of plane is
\[(x, y, z) - (1, 1, 1) \cdot (42, -42, 14) = 0\]

\[3x - 3y + z - 1 = 0\]

Ex: find eqns of the osc. circle of \(y = x^{2/3}\) at \((0, 0)\) and \((1, 1/2)\)

\[\begin{align*}
    x & = 0 \\
    t & = 1 \\
    (t = 0)
\end{align*}\]

require \(\vec{N}\) and \(\vec{k}\) at each point.

\[\frac{1}{\sqrt{k}}\]

centers of the circle will be located at

\[\begin{align*}
    (0, 0) + \vec{N}(0) \frac{1}{\sqrt{k(0)}} \\
    (1, 1/2) + \vec{N}(1) \frac{1}{\sqrt{k(1)}}
\end{align*}\]
for \( k \), use

\[
k = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}
\]

\( k(0) = 1, \quad k(1) = \frac{1}{2\sqrt{2}} \)

for \( n \), first parameterize the curve

\[
\vec{r}(t) = (t, \frac{t^2}{2})
\]

\[
\vec{r}'(t) = (1, t) \quad \leftarrow \text{point is this direction}
\]

\[
\vec{T} = \frac{\vec{r}'(t)}{\sqrt{1+t^2}}
\]

\[
\vec{N} = \left( -t, 1 \right) \frac{\vec{N}}{\sqrt{1+t^2}}
\]

\[
\vec{N} \cdot \vec{T} \propto (1, t) \cdot (-t, 1) = 0
\]

why not \( \vec{N} = \left( t, -1 \right) \frac{\vec{N}}{\sqrt{1+t^2}} \)
\[ \mathbf{N} = \frac{(-t, 1)}{\sqrt{1 + t^2}}. \]

So at \((0, 0)\) \((t = 0)\)

\[ \mathbf{N} = (0, 1) \]

so center of circles located at

\[ (0, 0) + \frac{1}{2} (0, 1) = (0, 1) \]

so osc. circle is

\[ x^2 + (y-1)^2 = 1 \]

\[ r = (t, t^{2/3}) \]

at \((1, 1/2)\) \((t = 1)\)

\[ \mathbf{N} = \frac{(-1, 1)}{\sqrt{2}} \]

\[ \kappa = \frac{1}{2\sqrt{2}} \]

radius = \(2\sqrt{2}\)

center located at

\[ (1, 1/2) + 2\sqrt{2} \frac{(-1, 1)}{\sqrt{2}} = (-1, 5/2) \]
so osc. circle is

$$(x+1)^2 + (y-5/2)^2 = 8.$$