we will cover chapters 13, 16

Ch. 13 curves in 2 and 3 dimensions, their parameterizations, properties (e.g., tangents, curvature) and physical meanings

Ch. 16 parameterization of surfaces, integrals over curves and surfaces, analogs of Fundamental Theorem of Calculus in 2 and 3 dimensions

13.1 vectors and functions and space curves

scalar functions take in arguments (inputs) and output scalar value ∈ domain

\[ f(x) = x^2 \quad -\infty < x < \infty \]
\[ f(x) \geq 0 \quad \text{range} \]
\[ f(x, y) = \sqrt{1-x^2-y^2}, \quad x^2+y^2 \leq 1 \]

\[ 0 \leq f(x, y) \leq 1 \]

The range of a vector valued function (or vector function) is a set of vectors

\[ \vec{r}(t) = (t, t^2, \sqrt{1-t}) \]

\[ = t\vec{i} + t^2\vec{j} + \sqrt{1-t}\vec{k} \]

\[ \vec{i} = (1, 0, 0), \quad \vec{j} = (0, 1, 0), \quad \vec{k} = (0, 0, 1) \]

Domain: \( 1-t \geq 0 \quad 1 \geq t \)

Range: set of points that lie on some curve in 3D.

Each component of \( \vec{r}(t) \) is a scalar function.

\[ \left\{ f(t) = t, \quad g(t) = t^2, \quad h(t) = \sqrt{1-t} \right\} \]
limits of vector functions

If \( \mathbf{r}(t) = (f(t), g(t), h(t)) \),

\[
\lim_{t \to a} \mathbf{r}(t) = (\lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t)).
\]

\( \mathbf{r}(t) \) is continuous if \( \lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a) \).

\( \mathbf{r}(t) \) cont's iff each of its components in cont's.

as \( t \) is allowed to vary over the domain, \( \mathbf{r}(t) \) traces out a space curve \( C \). \( t \) is called the parameter, \( \mathbf{r}(t) \) (with domain specified) is called a parameterization of \( C \).

each value of \( t \) specifies some point on the curve.
Ex: \[ x = 1 - t, \quad y = 2 + 2t, \quad z = 3 + 4t \]

\[-\infty \leq t < \infty\]

\[ r = (x, y, z) = (1 - t, 2 + 2t, 3 + 4t) \]

\[ = (1, 2, 3) + t (-1, 2, 4). \]

This describes a straight line parallel to the vector \((-1, 2, 4)\) and passing through \((1, 2, 3)\).
Example 1:
\[ x = 2 \cos t, \quad y = 2 \sin t, \quad 0 \leq t < 2\pi. \]
\[ x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4 \]

Example 2:
\[ x = 2 \cos(t^2), \quad y = 2 \sin(t^2) \]
\[ 0 \leq t < \sqrt{2\pi}. \]
\[ (x^2 + y^2 = 4 \text{ still).} \]

Parameterizations are never unique.

Notice: 1 parameter traces out a 1-dimensional curve, 2 parameters trace out a 2-dimensional surface, 3 parameters trace out a 3-dimensional volume.
\[ \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + tk \mathbf{k}; \quad t \geq 0. \]

curve in 3D.

Since \( x^2 + y^2 = 1 \), the curve must lie on the cylinder \( x^2 + y^2 = 1 \). Also, as \( t \) increases, the \( z \) component increases linearly.

This curve is called a helix.
Ex: find a parameterization for the line segment starting at \( \vec{r}_0 = (1, 2, 3) \) and ending at \( \vec{r}_1 = (1, -1, 4) \).

\[ \vec{r}(t) = \vec{r}_0 + t (\vec{r}_1 - \vec{r}_0) \quad 0 \leq t \leq 1. \]

check: \( t = 0 \): \( \vec{r}(0) = \vec{r}_0 \)

\( t = 1 \): \( \vec{r}(1) = \vec{r}_1 \),

also \( \vec{r}(t) = \vec{r}_0 + (t-100) (\vec{r}_1 - \vec{r}_0) \)

\[ 100 \leq t \leq 101. \]

\[ \vec{r}(t) = \vec{r}_0 + \frac{1}{2} (t) (\vec{r}_1 - \vec{r}_0) \]

\[ 0 \leq t \leq 2. \]

\[ \vec{r}_1 - \vec{r}_0 = (0, -3, 1) \]

\[ \vec{r}(t) = (1, 2, 3) + t (0, -3, 1) \quad 0 \leq t \leq 1 \]

\[ \quad = (1, 2 - 3t, 3t + 2) \quad 0 \leq t \leq 1. \]
running $t$ from $1 \to 0$ traces this line out backwards.

\[ z = t \]

let $z = t$, $x^2 + y^2 = t$

\[
\begin{align*}
x &= \sqrt{t} \cos (4\pi t) \\
y &= \sqrt{t} \cos (4\pi t) \\
t &= t
\end{align*}
\]

$z$ increases linearly.

$0 \leq t \leq 1$

2 full revolution in $x$-$y$ plane as $t: 0 \to 1$. 
Ex at what points does helix $\mathbf{r}(t) = (\sin t \cos t, \sin t, t)$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

$\cos^2 t + \sin^2 t + t^2 = 5$

$t^2 = 4 \quad t = \pm 2$.

So points of intersection are

$\mathbf{r}(2) = (\cos 2, \sin 2, 2)$

$\mathbf{r}(-2) = (\cos 2, -\sin 2, -2)$

Ex two people walk along the paths according to

$\mathbf{r}_1(t) = (t, t^3, t^3) \quad t \geq 0.$

$\mathbf{r}_2(t) = (1+2t, 1+6t, 1+14t).$
q1: do these two people meet?
q2: will they visit common points?

q1: require: \[ t = 1 + 2t \]
\[ t^2 = 1 + 6t \]
\[ t^3 = 1 + 14t \]
\[ \rightarrow \text{cannot satisfy} \]

q2: find \( t_1, t_2 \) such that \( \mathbf{r}_1(t_1) = \mathbf{r}_2(t_2) \).

\[ t_1 = 1 + 2t_2 \quad , \quad t_1^2 = 1 + 6t_2 \quad , \quad t_1^3 = 1 + 14t_2 \]

\[ \Rightarrow \]
\[ (1 + 2t_2)^2 = 1 + 6t_2 \quad \Rightarrow \quad t_2 = 0, 1/2 \]
\[ \Rightarrow \quad t_1 = 1, 2 \]

3rd eqn? \[ t_1 = 1, t_2 = 0 : \quad 1 = 1 \quad \checkmark \]
\[ t_1 = 2, t_2 = 1/2 : \quad 8 = 8 \quad \checkmark \]

so the paths intersect at
\( \vec{r}_1 = (2, 4, 8) \) \hspace{1cm} (t_1 = 2) \\
\( \vec{r}_1' = (1, 1, 1) \) \hspace{1cm} (t_1' = 1)