This midterm has 5 questions on 10 pages, for a total of 45 points.

Duration: 75 minutes

• Write your name or your student number on every page.

• You need to show enough work to justify your answers.

• Continue on the back of the page if you run out of space.

• You have to turn in all 10 pages of this booklet even if you don’t use all the pages.

• This is a closed-book examination. None of the following are allowed: documents or electronic devices of any kind (including calculators, cell phones, etc.)

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1. Let

\[
A = \begin{pmatrix}
1 & 3 & 2 & -8 \\
5 & 15 & 6 & -32 \\
-1 & -3 & 2 & 0 \\
3 & 9 & 2 & -16
\end{pmatrix}.
\]

The row-reduced echelon form of \( A \) and \( A^T \) are given by

\[
A \sim \begin{pmatrix}
1 & 3 & 0 & -4 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad A^T \sim \begin{pmatrix}
1 & 0 & 4 & -2 \\
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

(a) Write down a basis for the column space of \( A \) denoted \( R(A) \).

(b) Write down a basis for the column space of \( A^T \) denoted \( R(A^T) \).

(c) Do the vectors \( \begin{pmatrix} 1 \\ 0 \\ 4 \\ -2 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \) form a basis for the column space of \( A \)?

Explain your answer. No part marks will be given here - both the answer and explanation have to be correct in order to receive any points.

(d) Find a basis for the null space of \( A \) denoted \( N(A) \).

(e) Let \( \{u_1, u_2\} \) be a basis for \( N(A) \) and let \( \{w_1, w_2\} \) be a basis for \( N(A^T) \). What condition(s) must \( b \) satisfy so that at least one solution to \( A^T x = b \) exists?

(f) Suppose there exists a vector \( x_p \) such that \( Ax_p = f \), where \( A \) is given as above. What is the set of all possible solutions to the system \( Ax = f \)?

(g) For any matrix \( M \), there is one and only one vector that lies both in \( R(M) \) and \( N(M^T) \). Show that this vector must be the \( 0 \) vector.

(h) Write the Matlab command(s) that produce(s) the row-reduced echelon form of the transpose of a matrix \( P \). Assume that \( P \) has already been defined.
b) Yes. They form a basis for the row space of $A^T$, which is the column space of $A$.

d) Look at $\text{rref}(A)$:

$$x_3 - 2x_4 = 0 \Rightarrow x_3 = 2x_4$$

$$x_1 + 3x_2 - 4x_4 = 0 \Rightarrow x_1 = -3x_2 + 4x_4$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 + 4x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

So a basis is $\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 5 \\ 2 \end{pmatrix} \right\}$

e) $b^T u_1 = 0$, $b^T u_2 = 0$ for $R(A^T) \perp N(A)$ both have to be satisfied simultaneously.

f) $x = x_0 + c_1 u_1 + c_2 u_2$
2. Consider the matrix

\[ A = \begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}. \]

(a) Let \( z = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \) with \( 0 \leq t < 2\pi \). Show that \( \|Az\|_2^2 \) (i.e., the square of the 2-norm of \( Az \)) can be written as

\[ \|Az\|_2^2 = a + b\sin(2t), \]

for some constants \( a \) and \( b \). Give values for \( a \) and \( b \). Hint: use the identity \( \sin(2t) = 2\sin t\cos t \).

(b) Recall that

\[ \|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|_2 = 1} \|Ax\|_2, \]

where \( \max_{\|x\|_2 = 1} \) denotes the search over all vectors \( x \) such that \( \|x\|_2 = 1 \). Find \( \|A\|_2 \).

(c) Let \( y = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \). Give an upper bound for \( \|Ay\|_2 \).

(d) [This part does not pertain to any of the parts above.] Draw the set of all points \( x \) for which \( \|x\|_\infty = 1 \). On the same figure, draw the set of all points for which \( \|x\|_1 = 2 \). Label the plots and also label some points for reference.

\[ a = 25, \quad b = -20 \]

b) the largest \( \|Ax\|_2^2 \) can be when \( \|x\|_1 = 1 \) is

\[ 45 \quad \text{(when } \sin 2t = -1) \quad \Rightarrow \quad \|A\|_2 = \sqrt{45}. \]
\begin{align*}
\|A y\|_2 & \leq \|A\|_2 \|y\|_2 = \sqrt{45} \sqrt{20} \\
& = 30
\end{align*}
3. Let\
\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/100 & 0 \\ 0 & 0 & 100 \end{pmatrix}.
\]

Below, we denote \( x_0 \) as the solution to \( Ay = b_0 \), and \( x_0 + \Delta x \) as the solution to \( Ay = b \).

2 marks (a) Compute the condition number \( \text{cond}(A) = \|A\|_2 \|A^{-1}\|_2 \).

2 marks (b) Suppose \( b_0 \) is fixed, and let \( b = b_0 + \Delta b \), where \( \|\Delta b\|_2 = 1 \). For what vector \( \Delta b \) is the ratio \( \|\Delta x\|_2 / \|x_0\|_2 \) as small as possible? For what vector \( \Delta b \) is the ratio \( \|\Delta x\|_2 / \|x_0\|_2 \) as large as possible? One point given for each correct answer.

2 marks (c) Suppose \( \|b_0\|_2 = 1 \) and \( \|\Delta b\|_2 = 1 \). For what \( b_0 \) would the ratio \( \|\Delta x\|_2 / \|x_0\|_2 \) be less than or equal to 1 for any \( \Delta b \) that lies on the unit sphere?

\[
a) \quad \|A\|_2 = \max \text{ of diagonal entries of } A = 100 \\
\|A^{-1}\|_2 = \max \text{ of diagonal entries of } A^{-1} = 100 \\
\text{cond} (A) = \|A\|_2 \|A^{-1}\|_2 = 1 \	imes 10^4
\]

b) \( b_0 \) fixed, so \( x_0 = A^{-1} b_0 \) is also fixed. The largest \( \frac{\|\Delta x\|_2}{\|x_0\|_2} \) ratio is achieved when \( \|\Delta x\|_2 \) is maximized, where \( \Delta x = A^{-1} \Delta b \). The largest stretch produced by \( A^{-1} \) is \( \|A^{-1}\|_2 = 100 \). This occurs when \( \Delta b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) (largest entry in \( A^{-1} \) since \( A \) is diagonal).
(a) The smallest stretch performed by $A^{-1}$ is equal to the smallest entry in $A^{-1}$ (since $A^{-1}$ is diagonal), which is $\frac{1}{100}$. This occurs when $\Delta b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\text{smaller stretch} = \frac{1}{\|A\|_2} = \frac{1}{100}$$

\[
\begin{align*}
\frac{\|\Delta x\|_2}{\|x_0\|_2} & \text{ is max when } \Delta b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\frac{\|\Delta x\|_2}{\|x_0\|_2} & \text{ is min when } \Delta b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{align*}
\]

c) If $\|\Delta b\|_2 = 1$, $\|\Delta x\|_2 \leq 100$ from above.

Let $b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $x_0 = \begin{pmatrix} 0 \\ 100 \end{pmatrix}$.

$\Rightarrow \|x_0\| = 100$. So

\[
\frac{\|\Delta x\|_2}{\|x_0\|} = \frac{\|\Delta x\|_2}{100} \leq 1
\]

\[
\text{so } b_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
4. Consider the boundary value problem

\[ \frac{d^2 u}{dx^2} + c \frac{du}{dx} + k(x)u = f(x), \quad u(0) = u(1) + A, \quad u'(0) = u'(1), \]

where \( c \) is a constant. Parts (a)-(d) below pertain to the above boundary value problem. Assume that the interval \([0, 1]\) is divided into \( N \) equally spaced points \( x_1, \ldots, x_N \) so that \( x_1 = 0 \) and \( x_N = 1 \) (the spacing between adjacent points is \( h \)). Denote the corresponding values of \( u \) at the discrete points as \( u_1, \ldots, u_N \).

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<td>(a) Use finite differences to write the differential equation in discrete form. You may use either one-sided differencing or central differencing for the ( du/dx ) term. <strong>Make sure to state the range of the index over which the discretization is valid.</strong> That is, if your index is ( j ), indicate the smallest and largest value of ( j ).</td>
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<tr>
<td>1</td>
<td>(b) Write the boundary condition ( u(0) = u(1) + A ) in terms of ( u_1 ) and ( u_N ).</td>
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<td>(c) Use one-sided differencing to write the boundary condition ( u'(0) = u'(1) ) in discrete form.</td>
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<td>(d) Suppose that, in Matlab, the variables ( A ) and ( f ) are defined. Write the Matlab command that returns the solution to the system ( Au = f ). You may assume that ( A ) is non-singular.</td>
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<td>(e) <strong>[This part does not pertain to any of the parts above.]</strong> Suppose you are given that ( v(x_1) = v_1, v(x_2) = v_2, ) and ( v(x_3) = v_3 ). You are also given that ( x_2 - x_1 = h_1 ) and ( x_3 - x_2 = h_2 ), where ( h_1 ) and ( h_2 ) are both positive and small but are not equal. Give a finite differences approximation to the second derivative of ( v ) at ( x_2 ) in terms of ( v_1, v_2, v_3, h_1, ) and ( h_2 ). The associated error should go to 0 as ( h_1 ) and ( h_2 ) both go to 0.</td>
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\[
a) \quad \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + \frac{u_{j+1} - u_{j-1}}{2h} + k(x_j)u_j = f(x_j) \\
\quad j = 2, \ldots, N-1 \\
\]

\[
b) \quad u_1 = u_N + A \\
c) \quad \frac{u_2 - u_1}{h} = \frac{u_N - u_{N-1}}{h} \\
\]
a) \[ u = A \setminus f \]

e) \[ v(x_1) = v(x_2 - h_1) \approx v(x_2) - h_1 v'(x_2) + \frac{h_1^2}{2} v''(x_2) \]

\[ v(x_3) = v(x_2 + h_2) \approx v(x_2) + h_1 v'(x_2) + \frac{h_2^2}{2} v''(x_2) \]

\[ h_2 v_1 + h_1 v_3 = h_2 v_2 + h_1 v_2 + h_2 h_1 \frac{v''(x_2)}{2} \]

\[ + \frac{h_1 h_2^2}{2} v''(x_2) \]

\[ h_2 v_1 - (h_1 + h_2) v_2 + h_1 v_3 = \left[ h_1 h_2^2 + h_2 h_1 \right] v''(x_2) \]

\[ v''(x_2) = \frac{h_2 v_1 - (h_1 + h_2) v_2 + h_1 v_3}{h_1 h_2 \left( \frac{h_1 + h_2}{2} \right)} \]
5. Short answer questions. All questions are separate from each other.

2 marks (a) Consider the following Matlab program:

```
clear all
x = [1, i, 2-i, -3];
y = x';
z = transpose(x);
```

What is the value of y? What is the value of z?

2 marks (b) Is the set of all singular $3 \times 3$ matrices a subspace? If yes, show that it is. If no, provide a counterexample.

2 marks (c) The set of all $3 \times 3$ symmetric matrices is a subspace. Find a basis for this subspace and also state the dimension of the subspace.

(d) [BONUS (2 marks)] The solution $u(x)$ to

$$u'' = -f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u(1) = 0,$$

describes the temperature distribution in a rod of unit length with energy input $f(x)$. The solution $u(x)$ to this equation is unique because the only solution to the homogeneous equation

$$v'' = 0, \quad 0 < x < 1, \quad v(0) = 0, \quad v(1) = 0,$$

is $v \equiv 0$. Explain the physical meaning of this statement. No part marks.

\[ a) \begin{pmatrix} y \\ \end{pmatrix} = \begin{pmatrix} 1 \\ -i \\ 2 + i \\ -3 \end{pmatrix}, \begin{pmatrix} z \\ \end{pmatrix} = \begin{pmatrix} 1 \\ i \\ 2 - i \\ -3 \end{pmatrix} \]

\[ b) N_o. \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

$A, B$ both singular. $A + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ not singular
c) A general $3 \times 3$ symmetric matrix can be written as

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$+ e \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So a basis is

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \text{ dimension 6}.$$
d) In this case, the only things that affect the steady state temperature distribution are 1) \( f(x) \), the external heat source 2) the initial condition.

Notice that \( v(x) \) is the steady state temperature resulting from any initial condition and in the absence of a heat source.

The fact that \( v \equiv 0 \) is the only solution says that there is no non-uniqueness associated with different initial conditions; all initial conditions lead to \( v \equiv 0 \). Essentially, the effects of the initial condition decay (exponentially) in time. The temperature therefore must be uniquely determined by the heat source.