How to calculate $x_{n+1}$ from $x_n$?

\[ x_{n+1} = \begin{pmatrix} x_{n+1,1} \\ x_{n+1,2} \\ x_{n+1,3} \end{pmatrix}. \]

\[ x_{n+1,1} = x_{n,1} p_{11} + x_{n,2} p_{12} + x_{n,3} p_{13} \]

\[ x_{n+1,2} = x_{n,1} p_{21} + x_{n,2} p_{22} + x_{n,3} p_{23} \]

\[ x_{n+1,3} = x_{n,1} p_{31} + x_{n,2} p_{32} + x_{n,3} p_{33} \]

\[
\begin{pmatrix} x_{n+1,1} \\ x_{n+1,2} \\ x_{n+1,3} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} x_{n,1} \\ x_{n,2} \\ x_{n,3} \end{pmatrix} \]

Col's sum to 1

Then given some initial state $x_0$, we have

\[ x_n = P^n x_0. \]

$P$ has 2 important properties
1) all entries are non-negative and less than or equal to 1

\[ 0 \leq p_{ij} \leq 1 \]

2) each column sum of the matrix \( P \) sums to 1

Matrices with these two properties are called stochastic matrices. Question: given an initial social group (specifies \( x_0 \)) where will the student end up (eventually)?

e.g.

\[
P = \begin{pmatrix}
0.75 & 0 & 0 \\
0.18 & 0.75 & 0.14 \\
0.18 & 0.14 & 0.34
\end{pmatrix}
\]

assume \( x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) (starts in group 1)

we expect \( x_n \to \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \end{pmatrix} \) as \( n \to \infty \).

Interpretation: eventually, he will spend half his time in group 2, half in group 3, and none in group 1.
what's happening?

P has the following eigenpairs:

\[ \lambda_1 = 1 \quad v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \]
\[ \lambda_2 = \frac{1}{2} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \]
\[ \lambda_3 = \frac{3}{4} \quad v_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} v_1 + 0 v_2 + \frac{1}{2} v_3 \]

\[ P^n x_0 = \frac{1}{2} P^n v_1 + \frac{1}{2} P^n v_3 \]
\[ = \frac{1}{2} \lambda_1^n v_1 + \frac{1}{2} \lambda_3^n v_3 \]

as \( n \to \infty \), \( \lambda_1^n \to 1 \)
\( \lambda_3^n \to 0 \)

\( \Rightarrow P^n x_0 \to \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \) as \( n \to \infty \)
umbrella problem (assignment 1)

the two relevant states are

1) he has umbrella
2) he has no umbrella.

\[ p_0 = 0 \]

\[ p_{10} = 1 \]

\[ p = p_1 \]  \( \text{prob. of rain is } p \)

\[ p_{01} = 1 - p \]

\[ \text{has no umbrella} \]

\[ \text{has umbrella} \]

Let

\[ x_{n,0} = \text{prob. he has no umbrella on } n^{th} \text{ trip} \]

\[ x_{n,1} = \text{prob. he does have an umbrella on the } n^{th} \text{ trip.} \]

\[ x_{n+1,0} = x_{n,0} p_{00} + x_{n,1} p_{01} \]

\[ x_{n+1,1} = x_{n,0} p_{10} + x_{n,1} p_{11} \]
\[ x_{n+1} = \begin{pmatrix} 0 & 1-p \\ b1 & p \end{pmatrix} x_n \]

\[ x_{n+1} = p^{n+1} x_0 \]

Eigenpairs:

\[ \lambda_1 = 1, \quad v_1 = \begin{pmatrix} 1-p \\ 1 \end{pmatrix} \]

\[ \lambda_2 = p-1, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Suppose \( x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

\[ x_0 = \frac{1}{2-p} v_1 + c_2 v_2 \]

\[ x_n = p^n x_0 = \frac{1}{2-p} p^n v_1 + c_2 p^n v_2 \]

\[ = \frac{1}{2-p} \lambda_1^n v_1 + c_2 \lambda_2^n v_2 \quad n \to \infty. \]

\[ \to \quad \frac{1}{2-p} \begin{pmatrix} 1-p \\ 1 \end{pmatrix} \]
So in the long run, the prob. that he is without an umbrella is \( \frac{1-p}{2-p} \).

The prob. that he gets wet is then

\[
P(\text{wet}) = \frac{p(1-p)}{2-p}
\]

\[
\frac{dp(\text{wet})}{dp} = 0 \rightarrow p = 2 - \sqrt{2} \approx 0.58
\]

### Properties of Stochastic Matrices

- All entries non-negative
- All entries can add up to 1

1) if \( \mathbf{x} \) is a state vector, then so is \( \mathbf{P}\mathbf{x} \). Look at p. 243

\[
x_{n+1,1} + x_{n+1,2} + x_{n+1,3} = x_{n,1} \left[ p_{11} + p_{21} + p_{31} \right] \\
+ x_{n,2} \left[ p_{12} + p_{22} + p_{32} \right] \\
+ x_{n,3} \left[ p_{13} + p_{23} + p_{33} \right]
\]
Since each column in $P$ sums to 1, we have

\[ p_{11} + p_{21} + p_{31} = 1 \]
\[ p_{12} + p_{22} + p_{32} = 1 \]
\[ p_{13} + p_{23} + p_{33} = 1 \]

\[ x_{n+1,1} + x_{n+1,2} + x_{n+1,3} = x_{n,1} + x_{n,2} + x_{n,3} = 1 \]

generlize to $n \times n$...

2) $P$ has an evec of 1

rows of $P^T$ sum to 1

\[ P^T \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) = \lambda \left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) \]

so $\lambda = 1$ is an evec of $P^T$ with evec $\left( \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right)$

\[ \det (P^T - I) = 0 \]
\[ \det ((P^T - I)^T) = 0 \]
\[ \det (P - I) = 0 \Rightarrow \lambda = 1 \text{ is an evec of } P. \]
3) the other eigenvalues satisfy $|\lambda_j| \leq 1$

use the fact that

$$\| P x_1 \|_1 \leq \| x_1 \|_1,$$

$\uparrow$ see p. 188 in online notes

suppose $P x_j = \lambda_j x_j$

$$\| P x_j \|_1 = |\lambda_j| \| x_j \|_1 \leq \| x_j \|_1,$$

$\Rightarrow |\lambda_j| \leq 1 \text{ for all } j = 1, \ldots, n$

4) the eigenvector $x$, corresponding to $\lambda_1 = 1$

has all non-negative entries.

consider the case where $\lambda_1 = 1$, $|\lambda_j| < 1$

for $j = 2, \ldots, n$

pick $x_0$ and let it be a stable vector.

then $x_n = p^k x_0$ will also be a

State vector
but the power method states that

\[ X \mathbf{v} \to c_1 \mathbf{v}_1 \quad \text{as} \quad k \to \infty. \]

\[ c_1 \mathbf{v}_1 \quad \text{also is a} \quad \text{state vector.} \]

Then \( \mathbf{v}_1 \) can also be chosen to have all non-negative entries.

\( \exists \mathbf{v}_2 \) more properties (no proof)

\( \exists \mathbf{v}_k \) if \( P \) or \( P^k \) has all positive entries

(a) \( \lambda_1 = 1 \), \( \mathbf{v}_1 \) can be chosen to have all positive entries

(b) all other \( \mathbf{v}_j \)'s satisfy \[ |\lambda_j| < 1 \]

\[ j = 2, \ldots, n. \]
Google Page rank
(not on exam)

when you google a phrase
e.g. "symmetric matrix", how does
Google return/find and rank the
results for you. There potentially
millions of pages that contain the
words "symmetric" and "matrix"
one way to rank the pages is to
let people do it. Problem: bias.

query "best USA presidents"

idea: let the web rank the pages.
The importance of a page is determined by the number of pages that link to that page and their importance. Call the importance of page \( i \): \( I(P_i) \)

Suppose \( P_j \) has \( l_j \) outgoing links.

If \( P_i \) is one of those links, then \( P_j \) passes on \( \frac{1}{l_j} \) of its importance to \( P_i \).

\[
I(P_i) = \sum_{P_j \text{ that link to } P_i} \frac{I(P_j)}{l_j}
\]

Let

\[
I = \begin{pmatrix}
  I(P_1) \\
  I(P_2) \\
  \vdots \\
  I(P_n)
\end{pmatrix}
\]

Then

\[
I = H I
\]

\( I \) is an eigenvector of \( H \) with eigenvalue 1.
H will be 0. This means that there are possibly many ears with one.

The problem is almost all entries in $I$ are 0 or 1 with some
pairs of $1$s and $0$s.

Ideally, use power method to compute $\mathbf{H}$.
further, some pages have no outgoing links.

\[ H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

\( H \) doesn't even have an edge of 1.

(2) is called a dangling node.

fix: for columns with all zeros, replace each entry by \( \frac{1}{n} \)

\[ S = H + A \]

\( A \) is all zeros except for col's corresponding to dangling nodes (these columns have all \( \frac{1}{n} \)).
$S$ is guaranteed to be a stochastic matrix.

Still not good enough

\[
S = \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

All eigenvalues of $S$ have abs. value 1. The power method will not converge.

Define
\[
G_\alpha = \alpha S + (1-\alpha) \frac{1}{n} \mathbf{1}, \quad 0 < \alpha < 1
\]

\[
1 = \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \right)
\]
in practice $\alpha = 0.85$

the second largest eve of $G_7$ is a

$\lambda_1^n \left[ - + \left( \frac{\lambda_2}{\lambda_1} \right)^n \ldots \right]$