last time, one sided difference

\[
f'(x) = \frac{f(x+h) - f(x)}{h} + \frac{h}{2} f''(x) + O(h^2)
\]

\[
\begin{array}{cccccc}
 & x_1 & x_2 & x_3 & \ldots & x_N \\
\hline
x & \downarrow & \downarrow & \downarrow & \ldots & \downarrow \\
y & \uparrow & \uparrow & \uparrow & \ldots & \uparrow \\
y_1, y_2, y_3, \ldots, y_N
\end{array}
\]

terms proportional to \( h^2 \) are and smaller

\[
f'(x_1) = \frac{y_2 - y_1}{2h} \quad f'(x_2) = \frac{y_3 - y_2}{h}
\]

In Matlab,

\[
y(2:end) = [y_2, y_3, \ldots, y_N]
\]

\[
y(1:end-1) = [y_1, y_2, \ldots, y_{N-1}]
\]

\[
\frac{1}{2}(y_{2:end}) - y_{1:end-1} = [y_2 - y_1, y_3 - y_2, \ldots, y_N - y_{N-1}]
\]

\[
\Delta x \quad \Delta x
\]
gives one-sided difference at \( x = x_1, x_2, \ldots, x_{N-1} \)

when \( f(x) \) is linear, this is exact

(error is 0)
a more accurate method is called centered difference

\[ f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4) \]
\[ f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + O(h^4) \]

\[ f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3} f'''(x) + O(h^5) \]

\[ f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{\frac{h^2}{3} f'''(x)}{2h} + O(h^5) \]

\[ y(3:end) - y(1:end-1) \]

\[ \frac{2\Delta x}{2\Delta x} \]

second derivative add (1) + (2).

\[ f''(x) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4) \]

\[ f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + O(h^2) \]
A gambler starts with $0 < x < 100$ dollars and keeps making bets until either he has $0$ or he has $100$. The probability $p(x)$ that he walks away with $100$ is given by

$$p'' - w p'(x) = 0$$

$$p(0) = 0$$

$$p(100) = 1$$

where $w < 0$ is a bias toward the casino.

Set up the system of equations for a finite difference solution for $p(x)$ (assume a uniform grid)

$$p_{i-1} - 2p_i + p_{i+1} = 0$$

$$x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_N$$

$$P_1 \quad P_2 \quad \ldots \quad P_N$$

Let $p(x_j) \approx p_j$ finite difference solution. We have $N$ unknowns $p_1, p_2, \ldots, p_N$ for $j = 2, 3, \ldots, N-2, N-1$.
\[ p(x_j - h) \xleftarrow{\text{p}(x_j)} p(x_j) \xrightarrow{\text{p}(x_j + h)} \]

\[
\frac{p_{j-1} - 2p_j + p_{j+1}}{h^2} + w \frac{p_{j+1} - p_{j-1}}{2h} = 0 \quad (j = 2, \ldots, N-1)
\]

(N-2) equations

The last two equations come from the boundary conditions:

\[ p_1 = 0 \quad p_N = 1 \]

\[ p(0) \quad p(100) \]

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & b_2 & b_3 & \cdots & b_{N-1} & b_N \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 1 \\
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
\vdots \\
p_N \\
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{pmatrix}
\]

\[
b_+ = \frac{1}{h^2} \\
b_- = -\frac{1}{h^2}
\]
\[ b_- = \frac{1}{h^2} - \frac{w}{2h} \]

\[ b_+ = \frac{1}{h^2} + \frac{w}{2h} \]

\[ u'' = f \]

\[ u(0) = u'(1) = 0 \]

If \( \tilde{u} \) is a solution, \( \tilde{u} + C \) is also a solution.

\[ u'(0) = A, \quad u'(1) = B. \]

When the value of \( u \) at the boundaries is specified, they are called Dirichlet conditions. When the value of \( u' \) is specified at the boundaries, they are called Neumann conditions.

\[ \text{for } j = 2, \ldots, N-1 \]

\[ u_{j-1} - 2u_j + u_{j+1} \]

\[ \frac{1}{h^2} - u_j = -1 \]

\[ \text{approaches for the boundary} \]

\[ 1) \text{ use one-sided difference at each boundary:} \]
\[ \frac{u_2 - u_0}{\Delta x} = A \quad (u'(0) = A) \]

\[ u_0 \quad \text{NOT } u_1 - u_0 \text{!!! sorry!} \]

\[ u_N - u_{N-1} = B \Delta x \]

\[ \text{NOT } -u_0 + u_1 \text{!!! sorry again...} \]

2) centered difference at the boundaries

(ghost point)

at the left boundary,

\[ \frac{u_2 - u_0}{2 \Delta x} = A \Rightarrow u'(0) = A \]

\[ u_0 = u_2 - 2A \Delta x \]

Next, use \( u_0 \) to compute the finite difference approx. for \( u \) at \( x = x_1 \) (which we previously could not do because we didn't have \( u_0 \))
instead of replacing \( \oplus \) by \( u''(0) \rightarrow u(0) \)
\[
\frac{u_0 - 2u_1 + u_1}{h^2} = -u_1 = -1
\]
with \( u_0 \) given as above,
\[
\frac{u_2 - 2A \Delta x \Delta x_1 - 2u_1 + u_2}{h^2} = -u_1 = -1
\]
similarly, replace \( \oplus \) by something similar.
using the ghost point \( x_{N+1} \)

Ch. 2 subspaces, bases, dimension

\( \mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots \) are examples of vector spaces. An element in each space is called a vector.
The vectors can also be functions. The space of all functions \( f(x) \) defined on \( 0 \leq x \leq 1 \) is also a vector space. Here, the "vectors" are functions.
within all vector spaces, two operations are defined.

a) addition of two vectors

b) multiplication of a vector by a scalar

they satisfy the following rules (let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) be vectors of a vector space \( V \)).

1) \( \mathbf{u} + \mathbf{v} \in V \)

2) \( \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \) (commutativity)

3) \( \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \)

4) exists a \( \mathbf{0} \) element such that \( \mathbf{u} + \mathbf{0} = \mathbf{u} \)

5) " an inverse element \( \mathbf{-u} \) s.t. \( \mathbf{u} + (-\mathbf{u}) = \mathbf{0} \)

6) if \( \mathbf{c} \) is a scalar, \( \mathbf{c} \mathbf{u} \in V \)

7) \( \mathbf{c} (\mathbf{u} + \mathbf{v}) = \mathbf{c} \mathbf{u} + \mathbf{c} \mathbf{v} \)

8) \( (\mathbf{c} + \mathbf{b}) \mathbf{u} = \mathbf{c} \mathbf{u} + \mathbf{b} \mathbf{u} \) (\( \mathbf{b} \) is a scalar as is \( \mathbf{c} \))

9) \( \mathbf{c} (\mathbf{b} \mathbf{u}) = (\mathbf{c} \mathbf{b}) \mathbf{u} \)

10) \( \mathbf{1} \mathbf{u} = \mathbf{u} \)