Matrix norms

\[ \| A \|_p = \max_{x \neq 0} \frac{\| A x \|_p}{\| x \|_p} \]

\[ \| A \|_\infty = \max \| A x \|_\infty \]

\[ p = 1, 2, \infty \ldots \]

Search over all vectors \( x \) s.t. \( \| x \|_p = 1 \), evaluate \( \| A x \|_p \), record, then take max

\[ \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \]

The set of all vectors \( x \) s.t. \( \| x \|_2 = 1 \)

can be written \( x = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \), \( 0 < \theta < 2\pi \)

\[ x_1^2 + x_2^2 = \cos^2 \theta + \sin^2 \theta = 1 \]

\[ A x = \begin{bmatrix} 5 \cos \theta & y_1 \\ 2 \sin \theta & y_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{y_1^2}{25} + \frac{y_2^2}{4} = 1 \]
\[ \| A(0) \|_2 = 5 \text{; it is the largest, therefore } \| A \|_2 = 5 \]

\[ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]  \[ \| A \|_{\infty} = ? \]

\[ b \]

\[ c \]

\[ d \]

\[ (1,1) \]

\[ (1,1) \]

\[ (1,0.1) \]

\[ (1,0.5) \]

\[ \| (1,0.1) \|_{\infty} = 1 \]

\[ \| (1,0.5) \|_{\infty} = 1 \]

\[ \text{how does } A \text{ map this square?} \]

\[ \text{segment } ab : \ y = 1, \ \text{max } -1 \leq x \leq 1 \]

\[ x = \begin{bmatrix} x \\ 1 \end{bmatrix} \]

\[ A \times = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \]
$\mathbf{A} \mathbf{x} = \begin{pmatrix} x + 2 \\ 3x + 4 \end{pmatrix}$

$\max ||A\mathbf{x}||_{\infty} = ?$

$\max_{-1 < x < 1} \begin{pmatrix} |x + 2| \\ |3x + 4| \end{pmatrix}$

achieves $\max$ at $x = 1$ where $\max = 7$.

Segment $bc$: $x = -1$, $-1 < y < 1$

$||A||_{\infty} = 7$

general formula for $||A||_{\infty}$ given in text

$||A||$, given in class notes

Condition number

$\text{Cond } (A) = ||A|| ||A^{-1}|| = \frac{\max \text{ stretch}}{\min \text{ stretch}}$

Condition number measures the sensitivity of the solution $\mathbf{x}$ to small changes in $b$ ($A\mathbf{x} = b$)
$\begin{bmatrix} 1 & 1 \\ 1.00001 & 1 \end{bmatrix} \Rightarrow \text{Cond} (A) = 4 \times 10^5$

let $b_0 = (1)$

$x_0 = A^{-1} b_0 = (0)$

$A b_0 = (1)$

$A x_0 = b_0$

let $\Delta b = (-0.1)$

let $b = b_0 + \Delta b$

notice $\frac{||\Delta b||}{||b_0||} = 0.1$ small relative change in $b$

$A x = b \Rightarrow x \approx \begin{pmatrix} -20000 \\ 20000 \end{pmatrix}$

$\Delta x = x - x_0 \approx \begin{pmatrix} -20000 \\ 20000 \end{pmatrix}$

$\frac{||\Delta x||}{||x_0||}$ is big large relative in solution
\[ \Delta b = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} \]

\[ b = b_0 + \Delta b = \begin{pmatrix} 1.1 \\ 1.1 \end{pmatrix}, \quad A \mathbf{x} = b \]

\[ \mathbf{x} = \begin{pmatrix} 1.1 \\ 0 \end{pmatrix} \]

\[ \Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0 = \begin{pmatrix} 1.1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.1 \\ 0 \end{pmatrix} \]

**cubic spline**

general idea of interpolation: given a set of data points \( (x_1, y_1), \ldots, (x_n, y_n) \), find a curve that passes through all data points and use it to provide an estimate for \( y \) for the value of \( y \) in between the data points.

Lagrange interpolation fits an order \( n-1 \) polynomial through the \( n \) data points. This method can become very sensitive when \( n \) large.
(Vandermonde is poorly conditioned when \( n \) large).

Basically, a high order polynomial can undergo many oscillations through minima/max which can be undesirable.

Cubic spline interpolation restricts the order to 3; however, the cubic function is different on each interval.

The coefficients in each interval are determined by continuity and differentiability conditions.

Use cubic spline interpolation to estimate the value of \( y \) at \( x = 1.5 \).
\[ f_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1 \]
\[ 0 < x < 1 \]
\[ f(x) = \begin{cases} 
  f_1(x) & 0 < x < 1 \\
  f_2(x) & 1 < x < 2 
\end{cases} \]

8 unknowns: \( a_1, b_1, \ldots, c_2, d_2 \)

\[ f_1(0) = 1 \Rightarrow \boxed{d_1 = 1} \]

\[ f_2(1) = 3 \Rightarrow \boxed{d_2 = 3} \]

\[ f_1(x) = a_1 x^3 + b_1 x^2 + c_1 x + 1 \]

\[ f_2(x) = a_2 (x-1)^3 + b_2 (x-1)^2 + c_2 (x-1) + 3 \]

\[ f(1) = 3 \Rightarrow f_1(1) = 3 \]

\[ a_1 + b_1 + c_1 + 1 = 3 \]
\[ a_1 + b_1 + c_1 = 2 \]

\[ f' \text{ is cont's at } x = 1 \]
\[ f_1' = 3a_1x^2 + 2b_1x + c_1 \]
\[ f_2' = 3a_2(x-1)^2 + 2b_2(x-1) + c_2 \]
\[ f_1'(1) = f_2'(1) \]
\[ \Rightarrow 3a_1 + 2b_1 + c_1 = c_2 \]
\[ 3a_1 + 2b_1 + c_1 - c_2 = 0 \]

\[ f'' \text{ is cont's at } x = 1 \]
\[ f_1'' = 6a_1x + 2b_1 \]
\[ f_2'' = 6a_2(x-1) + 2b_2 \]
\[ f_1''(1) = f_2''(1) \]
\[ \Rightarrow 6a_1 + 2b_1 = 2b_2 \]
\[ 6a_1 + 2b_1 - 2b_2 = 0 \]

\[ f_2(2) = 7 \]

\[ a_2 + b_2 + c_2 + 3 = 7 \]

\[ a_2 + b_2 + c_2 = 4 \]

\[ * \text{matlab uses different conditions...} \]

\[ f''(0) = 0 \]

\[ 2b_1 = 0 \]

\[ f''(2) = 0 \]

\[ 6a_2 + 2b_2 = 0 \]

\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & -1 \\
6 & 2 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 2 \\
\end{pmatrix}
\begin{pmatrix}
a_1 \\
b_1 \\
c_1 \\
a_2 \\
b_2 \\
c_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
0 \\
0 \\
4 \\
0 \\
0 \\
\end{pmatrix}
\]
Evaluate \( y \) at \( x = 1.5 \):

\[
y(1.5) = f_2(1.5)
\]

\[
= a_2 (1.5 - 1)^3 + b_2 (1.5 - 1)^2 + c_2 (1.5 - 1) + d_2
\]

\[
= 4.8125.
\]

Matlab command:

\[
\text{interp1}([0,1,2],[1,3,7],1.5,'spline')
\]

Finite differences

Idea: approximate derivatives using discrete points.

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

\[
\text{exact.}
\]

Finite differences:

\[
f'(x) \approx \frac{f(x+h) - f(x)}{h}
\]

for \( h \) "small"
this is called a one-sided difference
the error is proportional to $h$.

to see this,

\[
f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3 f'''(x)}{6} + \cdots
\]

$h$ small

\[
f(x+h) - f(x) = hf'(x) + \frac{h^2}{2} f''(x) + O(h^3)
\]

\[
\frac{f(x+h) - f(x)}{h} = \frac{1}{2} f''(x) + O(h^2) = f'(x)
\]

↑

↑

↑

↑

↑

an one-sided

difference

on a grid of $n$

points, can define

one-sided derivative

at $n-1$ points.