last time: given $n$ data points $(x_1, y_1), \ldots, (x_n, y_n)$
how do we find the values of $y$ between the
data points

**approach**: Lagrange interpolation

**goal**: find the lowest order polynomial that
passes through all $n$ data points, i.e.,
find $a_1, \ldots, a_n$ s.t.

$$p(x) = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

passes through all data points.

setting

$$p(x_i) = y_i \text{ for } i = 1, \ldots, n,$$

we have

$$
\begin{pmatrix}
    x_1^{n-1} & x_1^{n-2} & \ldots & 1 \\
    x_2^{n-1} & x_2^{n-2} & \ldots & 1 \\
    \vdots & \vdots & \ddots & \vdots \\
    x_n^{n-1} & x_n^{n-2} & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_n
\end{pmatrix}
=
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n
\end{pmatrix}
$$

$n \times n$ Vandermonde matrix
det ≠ 0 as long as \( x_1, \ldots, x_n \) are distinct.

\[ 2 \times 2 \]
\[ \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} = -x_2 + x_1 \]

\[ 3 \times 3 \]
\[
\begin{vmatrix}
 x_1^2 & x_1 & 1 \\
 x_2^2 & x_2 & 1 \\
 x_3^2 & x_3 & 1
\end{vmatrix}
= \begin{vmatrix}
 C_1 - x_1 C_2 \\
 C_2 - x_1 C_3 \\
 C_3 - x_1 \end{vmatrix}
= \begin{vmatrix}
 0 & x_2 - x_1 & 1 \\
 x_2^2 - x_1 x_2 & x_2 - x_1 & 1 \\
 x_3^2 - x_1 x_3 & x_3 - x_1 \end{vmatrix}
\]

\[ = \begin{vmatrix}
 x_2^2 - x_1 x_2 & x_2 - x_1 \\
 x_3^2 - x_1 x_3 & x_3 - x_1
\end{vmatrix}
= (x_2 - x_1)(x_3 - x_1)
\]

\[ = (x_2 - x_1)(x_3 - x_1)(x_2 - x_3) \]
induction ...
\[ \det (n \times n \text{ Vandermonde}) = (-1)^{\frac{n(n-1)}{2}} \prod_{i>j} (x_i - x_j) \]
always nonsingular when \( x_i \)'s are distinct.

**cubic splines** (approach 2).

![Diagram of cubic splines](image)

ducks placed at known data points, and the interpolation is performed by fitting a bendable but stiff strip around the ducks, and then letting it relax (it relaxes in a way that minimizes bending energy

\[ E = \int_{x_{i-1}}^{x_i} k^2 \, ds \]

\[ k^2 = \frac{\left( \int f'' \right)^2}{(1 + (f')^2)^{3/2}} \]
The cubic spline is essentially a mathematical model of the strip and reproduces the strip's interpolation.

Goal: find $f(x)$ that passes through all data points and has cont's first derivative and passes through minimizes $E$.

\begin{align*}
\text{Condition 1} & \quad f(x) \text{ is cont's and} \\ & \quad f(x_i) = y_i \quad \text{for } i = 1, \ldots, n
\end{align*}

\begin{align*}
\text{Condition 2} & \quad f'(x) \text{ exists and is cont's everywhere including at the duck points} \\
& \quad \text{("bendable but stiff")}
\end{align*}

\begin{align*}
\text{Condition 3} & \quad \text{higher order derivatives all exist in between the duck points} \\
& \quad \text{(allow possibility that they are discont's)}
\end{align*}
\[ E(f) = \int_{x_1}^{x_n} [f''(x)]^2 \, dx \text{ is minimized} \]

\text{a functional} \quad \text{(takes in a function, spits out a number)}

\[ \text{will yield 3 more conditions.} \]

\text{turns out that } f(x) \text{ is a piecewise cubic polynomial. Why?} \]

need to find } f(x) \text{ that minimizes } E \text{.}

\[ \text{if } x \text{ is a local minimum of } g(x), \text{ then } \frac{d}{dx} g(x + \varepsilon h) \bigg|_{\varepsilon = 0} = 0 \text{ for any } h \]

\[ \varepsilon h \left. g'(x + \varepsilon h) \right|_{\varepsilon = 0} = 0 \]

\[ \left. h g'(x) \right|_{\varepsilon = 0} = 0 \]

\[ g'(x) = 0 \text{ since } h \text{ arbitrary} \]
same approach for \( E(f) \):
suppose \( f(x) \) and \( f(x) + \varepsilon h(x) \) satisfy conditions 1, 2, 3 where \( f(x) \) minimizes \( E(f) \).

\[
f(x_i) = y_i \quad i = 1, \ldots, n
\]

\[
f(x_i) + \varepsilon h(x_i) = y_i
\]

\( \Rightarrow h(x_i) = 0 \quad i = 1, \ldots, n \)

\( h' \) is continuous everywhere

for simplicity, use \( n = 3 \).

\[
\frac{d}{d\varepsilon} E(f + \varepsilon h) \bigg|_{\varepsilon = 0} = 0 \quad \text{for any } h
\]

\[
\frac{d}{d\varepsilon} \int_{x_1}^{x_3} \left[ f''(x) + \varepsilon h''(x) \right]^2 dx \bigg|_{\varepsilon = 0} = 0
\]

\[
\int_{x_1}^{x_3} 2 h'' h'' dx \bigg|_{\varepsilon = 0} = 0
\]

\[
\int_{x_1}^{x_3} h'' f'' dx = 0 = \int_{x_1}^{x_2} h'' f'' dx + \int_{x_2}^{x_3} h'' f'' dx = 0
\]
Integrate by parts twice:

\[ h' f'' \big|_{x_1}^{x_2} - \int_{x_1}^{x_2} h' f''' \, dx + h' f'' \big|_{x_2}^{x_3} - \int_{x_2}^{x_3} h' f''' \, dx = 0 \]

Again:

\[ f'' h' - h f''' \big|_{x_1}^{x_2} + \int_{x_1}^{x_2} h f''' \, dx \]

\[ + f'' h' - h f''' \big|_{x_2}^{x_3} + \int_{x_2}^{x_3} h f''' \, dx = 0 \]

\[ h = 0 \text{ at } x_1, x_2, x_3 \]

So:

\[ h'(x_2) f''(x_2^-) - h'(x_1) f''(x_1^+) \]

\[ + h'(x_3) f''(x_3^-) - h'(x_2) f''(x_2^+) \]

\[ + \int_{x_1}^{x_2} f'''' h \, dx + \int_{x_2}^{x_3} f'''' h \, dx = 0 \]

\[ - h'(x_1) f''(x_1^+) + h'(x_2) [f''(x_2^-) - f''(x_2^+)] \]

\[ + h'(x_3) f''(x_3^-) + \int_{x_1}^{x_2} f'''' h \, dx + \int_{x_2}^{x_3} f'''' h \, dx = 0 \]
h is arbitrary so we can pick any function for h as long as h > 0 at x₁, x₂, x₃, and h' is cont's

1) if h ≡ 0 everywhere except on (x₁, x₂) also make h'(x₁) = h'(x₂) = h'(x₃) = 0

⇒ from (2) with h'(x₁) = h'(x₂) = h'(x₃) = 0 we have ∫ₙ₃₋ₙ₉ = 0 since h = 0 on (x₂, x₃)

⇒ ∫ₙ₁₋ₙ₉ h[f'''] / dx = 0

⇒ f''' = 0 since h arbitrary

⇒ f must be a cubic polynomial on (x₁, x₂)
do some thing except $h \equiv 0$ on $(x_i, x_2)$

$\Rightarrow f$ is a cubic on $(x_2, x_3)$

condition a. $f$ is a cubic (not necessarily that same cubic) on each interval $(x_i, x_{i+1})$.

so now $\Theta$ becomes.

$h'(x_1)f''(x_1^+) + h'(x_2)\left[f''(x_2^-) - f''(x_2^+)\right]$

$+ h'(x_3)f''(x_3^-) = 0$

choose $h$ to be

\[ h' = 1 \quad h' = 0 \quad h' = 0 \]

$\Rightarrow f''(x_1^+) = 0$

choose $h$ to be

\[ h' = 0 \quad h' = 0 \quad h' = 1 \]

$\Rightarrow f''(x_3^-) = 0$
choose \( h \) to be

\[ h' = 0 \quad h' = -1 \quad h' = 0 \]

\[ x_1 \quad x_2 \quad x_3 \]

\[ \Rightarrow f''(x_2^-) = f''(x_2^+) \]

\[ \Rightarrow f'' \text{ continuous everywhere including at the cusp points.} \]

Condition b

\[ f'' \rightarrow 0 \text{ at end points} \]

Condition c

\[ f'' \text{ cont's everywhere including at data points.} \]
with \( n \) data points, we have \( n-1 \) intervals \( \Rightarrow n-1 \) cubic polynomials
\( \Rightarrow 4(n-1) \) unknown coeff. to solve for.
conditions 1, 2, b, c will yield 4(n-1) unique equations to uniquely fix the coeff. of each cubic

\[
\begin{align*}
   &x_1 \quad x_2 \quad x_3 \quad x_4 \quad \ldots \quad x_{n-1} \quad x_n \\
   &p_1(x) \quad p_2(x) \quad p_3(x) \quad \ldots \quad p_{n-1}(x) \quad p_n(x)
\end{align*}
\]

\[
p_i(x) = a_i (x-x_i)^3 + b_i (x-x_i)^2 + c_i (x-x_i) + d_i
\]

instead of \( a_i x^3 + b_i x^2 + c_i x + d_i \)

\textbf{Condition 1} \quad p_i(x_i) = y_i \quad \text{for} \quad i = 1, \ldots, n-1

\( \Rightarrow \) \[d_i = y_i \quad \text{for} \quad i = 1, \ldots, n-1\] \((n-1)\) equations
\[ p_i(x_{i+1}) = y_{i+1}, \quad i = 1, \ldots, n-1 \]

\[ a_i(x_{i+1} - x_i)^3 + b_i(x_{i+1} - x_i)^2 + c_i(x_{i+1} - x_i) + d_i = y_{i+1}, \quad i = 1, \ldots, n-1 \]

\[ n-1 \text{ equations.} \]

**Condition 2**: (continuity of \( f' \) at data points)

\[ p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}), \quad i = 1, \ldots, n-2 \]

(no such conditions at \( x_1 \) and \( x_n \))

\[ 3a_i(x_{i+1} - x_i)^2 + 2b_i(x_{i+1} - x_i) + c_i = c_{i+1} \]

\[ p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}), \quad i = 1, \ldots, n-2 \]

\[ p_{i+1}(x) = a_{i+1}(x - x_{i+1})^3 + b_{i+1}(x - x_{i+1})^2 \]

\[ + c_{i+1}(x - x_{i+1}) + d_{i+1} \]

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