node could represent an object of a certain temperature (or voltage)

edge could represent conductors allowing heat flow (or current flow)

arrows indicate the assumed direction of flow

incidence matrix indicates how the nodes are connected
$D = \begin{pmatrix}
1 & 2 & 3 & 4 \\
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & -1 & 0 & 1 \\
1 & 0 & 0 & -1 \\
\end{pmatrix}$  \leftarrow \text{nodes}

\text{convention:}
\begin{align*}
in & : +1 \\
out & : -1
\end{align*}

what do the nullspace and column space of $D$ represent?

let $V = \begin{pmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
\end{pmatrix}$  
$v_j$ is the temperature (or voltage) at node $j$

$D \cdot V = \begin{pmatrix}
v_2 - v_1 \\
v_3 - v_2 \\
v_4 - v_3 \\
v_4 - v_2 \\
v_1 - v_4 \\
\end{pmatrix} = \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
\end{pmatrix} = \mathbf{b}$

$N(D)$:
this is the set of temperatures (or voltages) for which the temperature difference between all connected nodes is 0.
Looks like 5 equations for 4 variables, but in fact we really only have 3 constraints.

$V \in \mathbb{R}^4$ 3 constraints in $\mathbb{R}^4$ give a nullspace of dimension 1.

$N(D) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$  \quad \text{dim } N(D) = 1.

A graph may have a few disconnected pieces.

\[ D_1 = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & -1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & -1 & -1 \\
\end{pmatrix} \quad \text{dim } R(D_1) = 4 \]
\[ N(D) = \text{span} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

i.e., temperatures \( V_1, V_2, V_3 \) are equal
\( V_4, V_5, V_6 \) are equal
but the temp. in the left piece need not
be the same as that in the right piece.
in general, \( \dim N(D) \) is equal to the
number of isolated pieces.

\[ R(D) \]
represents the set of all possible temperature
differences. Can the differences be anything
we want?

we know
\[ \dim R(D) + \dim N(D) = 4 \]
(1) + (2) + (3) + (5) :

\[
\sqrt{2} - v_1 + (\sqrt{3} - v_2) + (v_4 - v_3) + (v_1 - v_4) = 0
\]

\[
\begin{align*}
\begin{array}{cccc}
& b_1 & b_2 & b_3 & b_5 \\
\end{array}
\end{align*}
\]

\[
b_1 + b_2 + b_3 + b_5 = 0
\]

(2) + (3) - (4)

\[
(\sqrt{3} - v_2) + (v_4 - v_3) - (v_4 - v_2) = 0
\]

\[
\begin{align*}
\begin{array}{cccc}
& b_2 & b_3 & b_4 \\
\end{array}
\end{align*}
\]

\[
b_2 + b_3 - b_4 = 0
\]

so \( b \) is constrained by

(A) \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} b = 0

(B) \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix} b = 0

(C) \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & -1 & 0
\end{bmatrix} b = 0

R(D) \perp N(D^T)
\[ \dim R(D) = 3 \]

so \( R(D) \) is a subspace of dimension 3 in \( \mathbb{R}^5 \). So it must be constrained.

how? The sum of temperature differences around a loop is 0:

recall \( Dv = \begin{pmatrix} v_2 - v_1 \\ v_3 - v_2 \\ v_4 - v_3 \\ v_4 - v_2 \\ v_1 - v_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix} = b \)

3 loops:

\( 0 + (4) + (5) \):

\[ (v_2 - v_1) + (v_4 - v_2) + (v_1 - v_4) = 0 \]

\[ b_1 + b_4 + b_5 = 0 \]

this looks like 3 constraints but

\[ (B) = (A) + (C) \] so effectively 2 constraints.
so $R^5 + 2$ constraints give a $3D$ 3-dimensional space (as expected).

What about $N(D^T)$ and $R(D^T)$?

\[
y_j = \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5
\end{pmatrix}
\]
interpret $y_j$ as the amount heat (or current) flowing in edge $j$

\[
D^T y_j = \begin{pmatrix}
y_5 - y_1 \\
y_1 - y_2 - y_4 \\
y_2 - y_3 \\
y_3 + y_4 - y_5
\end{pmatrix}
\]
\leftarrow the jth component is the heat flux into the jth node.

\[
N(D^T)
\]  
So $N(D^T)$ consists of all possible heat flows such that the net heat flux into each node is 0. Can we guess $N(D^T)$?
suppose the flows in edges 2 and 3 are both 0. ($y_2 = y_3 = 0$) The currents in edges 4, 5, 1 must be equal (what goes in must go out).

$$y_1 = y_4 = y_5$$

$$y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in N(D^T)$$

Similarly, take $y_1 = y_5 = 0$

$$y_2 = y_3 = -y_4$$

$$y = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in N(D^T)$$

Take $y_4 = 0$

$$y_1 = y_2 = y_3 = y_5 = 0$$

$$y = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in N(D^T).$$
\[ \dim N(D^T) = 2 \]
\[ \dim N(D^T) + \dim R(D^T) = 5 \]
\[ \uparrow \quad \uparrow \]
\[ 2 \quad 3 \]

Looks like we have 3 basis vectors for \( N(D^T) \), but

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
- \begin{pmatrix}
0 \\
1 \\
\vdots \\
0
\end{pmatrix}.
\]

So indeed only 2.

Another way to look at this:

If \( b \in R(D) \), then \( b \perp N(D^T) \)

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix},
\begin{pmatrix}
1 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
\vdots \\
0
\end{pmatrix}
\]

So these vectors must lie in \( N(D^T) \).
\[
R(D^T): \quad D^T y = x
\]

\[
D^T y = \begin{pmatrix}
  y_5 - y_1 \\
  y_1 - y_2 - y_4 \\
  y_2 - y_3 \\
  y_3 + y_4 - y_5
\end{pmatrix} = \begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix} = x
\]

\[x \in \mathbb{R}^4 \quad \text{dim } R(D^T) = 3\]

So, 1 constraint

What is the constraint?

Since any heat flow into a node had to come from another node (or nodes), the sum of all net heat flows into the nodes must be 0:

\[x_1 + x_2 + x_3 + x_4 = 0.\]

Another way to look at it:

\[R(D^T) \perp \text{span } \{1, 1, 1, 1\}.\]
So $\mathbf{x}^\top \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

**Ch. 3 Orthogonality**

Projections onto line

$L$ is the line through the origin passing through $\mathbf{a}$, i.e., $L = \text{span}\{\mathbf{a}\}$

$p$ is the projection of $\mathbf{x}$ onto $L$. It is the closest point to $\mathbf{x}$ on $L$.

Notice that $p$ can be written as a l.c. of the vectors that span $L$. So

$$p = s \mathbf{a}$$
The constant \( s \) is to be found by minimizing the distance \( \| x - sa \| \) from \( p \) to \( x \). The distance is minimized when the angle between \( x - p \) and \( a \) is \( \pi/2 \).

\[
a^T \left( sa - x \right) = 0
\]

\[
s a^T a - a^T x = 0
\]

\[
s = \frac{a^T x}{a^T a} = \frac{a^T x}{\|a\|_2^2}
\]

\( \Rightarrow \) the projection of \( x \) onto \( L \) is \( sa \).

\[
p = sa = \frac{a^T x}{\|a\|_2^2} a
\]

Can also obtain this result from regular calculus.
the square distance from $x$ to $p$ is

$$h^2 = (p - x)^T(p - x)$$

$$h^2 = (sa - x)^T(sa - x)$$

$$= s^2a^Ta - 2sa^Tx + x^Tx$$

choose $s$ s.t. $h^2$ is minimized

$$\frac{dh^2}{ds} = 2sa^Ta - 2a^Tx = 0$$

$$s = \frac{a^Tx}{a^Ta}$$

so now

$$p = \frac{a^Tx}{\|a\|^2}a$$

$$= \frac{a(a^Tx)}{\|a\|^2} = \frac{(a,a^T)x}{\|a\|^2}$$
\[ P = \left( \frac{a \ a^T}{||a||^2} \right) \times \text{projection matrix } P \]

If we are in \( \mathbb{R}^2 \), \( a \) is \( 2 \times 1 \).

\( a^T \) is \( 1 \times 2 \).