last, started talking about the four fundamental subspaces of a matrix \( A \):

- column space \( \text{Col}(A) \) or \( \text{Col}(A) \)
- set of all vectors that are l.c.'s of the columns of \( A \).

nullspace \( \text{N}(A) \) or \( \text{Nul}(A) \)

- the set of all vectors \( \mathbf{x} \) s.t. \( A\mathbf{x} = \mathbf{0} \)

the other two spaces are \( \text{R}(A^T) \), \( \text{N}(A^T) \).

finding a basis for \( \text{N}(A) \)

**Ex** find a basis for \( \text{N}(A) \) where

\[
A = \begin{pmatrix}
1 & 3 & 4 & 5 & 1 \\
2 & 2 & 2 & 2 & 2 \\
1 & 2 & 2 & 4 & 7
\end{pmatrix}
\]

find all solutions of \( A\mathbf{x} = \mathbf{0} \) \( \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \)
matlab \hspace{1cm} \text{rref} \left( \mathbf{A} \right) = \mathbf{U}

\begin{array}{ccc|ccc}
1 & 0 & 0 & -2 & -5 & U \\
0 & 1 & 0 & 5 & 18 & \leftarrow \\
0 & 0 & 1 & -2 & -12 & \end{array}

2 \hspace{1cm} \text{free variables}

\text{row 3:} \hspace{1cm} x_3 = 2x_4 + 12x_5

\text{row 2:} \hspace{1cm} x_2 = -5x_4 - 18x_5

\text{row 1:} \hspace{1cm} x_1 = 2x_4 + 5x_5

\mathbf{X} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} = \begin{pmatrix}
2x_4 + 5x_5 \\
-5x_4 - 18x_5 \\
2x_4 + 12x_5 \\
x_4 \\
x_5
\end{pmatrix} =

x_4, x_5 \hspace{1cm} \text{are free}

so the solutions to \hspace{1cm} \mathbf{A} \mathbf{X} = \mathbf{0} \hspace{1cm} \text{are all possible}

l.c.'s of \{ \mathbf{u}, \mathbf{v} \}. \hspace{1cm} \text{so} \hspace{1cm} \{ \mathbf{u}, \mathbf{v} \} \hspace{1cm} \text{certainly span the}
null space. But are they l.i.? to show this, put \( u \) and \( v \) in matrix

\[
\mathbf{A} = \begin{pmatrix}
2 & 5 \\
-5 & -18 \\
2 & 12 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[\mathbf{c} \]

and solve \( \mathbf{C} \mathbf{y} = 0 \)

\[
\begin{pmatrix}
2 & 5 \\
-5 & -18 \\
2 & 12 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix} = 0
\]

row 5: \( y_2 = 0 \)

row 4: \( y_1 = 0 \)

so \( \{u, v\} \) indeed are l.i. so forms a basis for \( N(\mathbf{A}) \)

notice: \( \dim N(\mathbf{A}) = 2 = \# \) of nonpivot columns in \( \mathbf{U} \).
Matlab: \( \text{null}(A) \rightarrow \text{get orthogonal basis for } \text{N}(A) \)

\( \text{null}(A, 'r') \rightarrow \text{get rational basis} \)

Recall that we ended up solving \( UX = 0 \) instead of \( AX = 0 \). Why is \( \text{N}(A) = \text{N}(U) \)?

Recall that row operations are performed by invertible elementary matrices \( E_j \) so that

\[
E_n E_{n-1} \ldots E_2 E_1 A = U
\]

\[
L^{-1}
\]

\[
\det L^{-1} = \det (E_n \ldots E_2 E_1) = \det E_n \det E_{n-1} \ldots \det E_1
\]

\( \neq 0 \)

So \( L^{-1} \) is invertible.

\[
L^{-1} A = U
\]

\[
A = LU
\]

So if we try to solve \( AX = 0 \)

\[
AX = LUx = 0
\]
\[ \mathbf{Ux} = L^{-1} \mathbf{0} \Rightarrow \mathbf{Ux} = \mathbf{0} \]

So \( A \mathbf{x} = \mathbf{0} \) if \( \mathbf{Ux} = \mathbf{0} \)

Finding a basis for the column space.

In general, \( R(A) \) is not the same as the \( R(LU) \).

E.g. \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) \( R(A) = \text{span} \{ (1) \} \).

\[ \mathbf{U} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \] \( R(LU) = \text{span} \{ (1) \} \).

Let \( \mathbf{U} = (u_1, u_1 a, u_1 b, \ldots, u_2, u_2 a, u_2 b, \ldots, u_k, \ldots, u_k) \) \( L^{n \times n} \)

\[ \mathbf{A} = LU = (Lu_1, Lu_1 a, Lu_1 b, \ldots, Lu_2, Lu_2 a, Lu_2 b, \ldots, Lu_k, \ldots, Lu_k) \] \( \mathbf{n} \times \mathbf{m} \) \( \mathbf{n} \times \mathbf{m} \) invertible.

However, if \( \{u_1, \ldots, u_k\} \) forms a basis for \( R(U) \), then the corresponding columns of \( A \).
\{L y_1, L y_2, \ldots, L y_n\} forms a basis for \(R(A)\), why? Need to show it is l.i. set, and that it span \(R(A)\).

1) L.i. if \(\{u_1, \ldots, u_k\}\) is a l.i. set, and \(L\) is invertible, we showed last time that \(\{L u_1, \ldots, L u_k\}\) is also a l.i. set.

2) show \(\bigvee\) span \(R(A)\).

Since \(\{u_1, \ldots, u_k\}\) forms a basis for \(R(U)\), we can write any vector in \(R(U)\)

\[
U \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = d_1 u_1 + \cdots + d_k u_k
\]

\[
A \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = d_1 L u_1 + \cdots + d_k L u_k
\]

For any vector in \(R(A)\)
so any vector in \( R(A) \) can be written as a l.c. of \( \{Lyu, \ldots, Lu_n\} \).

\[ \Rightarrow \text{spans } R(A) \]

1) + 2) \( \{Lyu, \ldots, Lu_n\} \) forms a basis for \( R(A) \).

Which columns of \( A \) form a basis for \( R(AU) \)? The pivot columns.

Conclusion: columns of \( A \) corresponding to pivot columns of \( U \) form a basis of \( R(A) \).

\[
EY \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 & 14 \\ -3 & -6 & -9 & 2 & -7 \\ -2 & -4 & -6 & 0 & 7 \end{bmatrix}
\]

\[
\sim \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in U
\]

\[
U_1 \quad U_1a \quad U_1b \quad U_2 \quad U_2a
\]
So a basis for \( R(A) \) is
\[
\{ \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 10 \end{pmatrix} \}.
\]

Notice: \( \dim R(A) = 2 = \# \text{ of pivot columns in } U \).

\text{Matlab}
\[
colspace(\text{sym}(A))
\]
returns some l.c. of what linear combination? HW.

\text{Rank: Let } A \text{ be } n \times m \text{ and } U = rref(A).
\[
\text{rank}(A) = \dim R(U) = \# \text{ of pivot columns in } U
\]
\[
\dim N(A) = \# \text{ of non pivot columns in } U
\]
\[
\text{rank}(A) + \dim N(A) = m \Leftrightarrow \# \text{ of column in } A.
\]

called the rank-nullity theorem
Basis for $R(A^T)$

one way: \( \text{rref}(A^T) \rightarrow \text{take pivot columns in } A^T \)

column space of $A^T$ or row space of $A$

the column space of $A^T$ is the set of all vectors $y$ s.t.

\[
y = A^T x
\]

but recall that \( A = LU \rightarrow A^T = U^T \overline{(L^T)} \)

\[
y = A^T x = U^T \overline{(L^T)} x = U^T z
\]

$\Rightarrow$ any vector in $R(A^T)$ (a d.c. of cols of $A^T$) can be written as a d.c. of the columns of $U^T$.

also

\[
y = U^T x = A^T (L^T)^{-1} x = A^T z
\]

$\Rightarrow$ any vector that is a d.c. of columns of $U^T$ can be written as a d.c. of the cols of $A^T$. 
So the columns of $U^T$ and cols of $A^T$ span the same space. This means that a basis for $R(U^T)$ is also a basis of $\text{R}(A^T)$.

What is a basis for $R(U^T)$? They are the l.i. cols of $U^T$, which are the l.i. rows of $U$. But all nonzero rows of $U$ are l.i. (because they all have pivots).

Example $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 14 \\ -3 & -6 & -9 & 2 & -7 \\ -2 & -4 & -6 & 10 & 17 \end{pmatrix}$

$U = \begin{pmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

A basis for $R(A^T)$ is

\[ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 5 \end{pmatrix} \]

Clearly l.i.
every nonzero row in $A$ has one and only one pivot.

$\Rightarrow \text{ rank } (A^T) = \dim \text{ R } (A^T) = \# \text{ nonzero rows in } A$

$= \# \text{ of pivots in } A$

$= \text{ Rank } (A)$

"Column rank = row rank"

Basis for $N(A^T)$ $A$ is $n \times m$

first, from before,

$\text{ rank } (A) + \dim \text{ Nul } (A) = m$

$\text{ rank } (A^T) + \dim \text{ Nul } (A^T) = n$

$\Rightarrow \dim \text{ Nul } (A^T) = n - \text{ rank } (A^T)$.

how to find basis for $N(A^T)$?

just row reduce $A^T$. . .
orthogonality.

for \( \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \), \( \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \)

the inner product (or dot product) is

\[
\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = (x_1, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \ldots + x_n y_n.
\]

remarks

1) \( \mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x} \)

2) \((c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2) \cdot \mathbf{y} = c_1 \mathbf{x}_1 \cdot \mathbf{y} + c_2 \mathbf{x}_2 \cdot \mathbf{y} \)

3) the 2-norm of a vector can be written in terms of the dot product

\[
\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\mathbf{x}^T \mathbf{x}}
\]

4) Cauchy–Schwarz inequality

\[
|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \quad \Rightarrow \quad -1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} \leq 1.
\]
5) the angle between \( x \) and \( y \) is then defined as

\[
\cos \theta = \frac{x \cdot y}{\|x\| \|y\|}
\]

\[-1 \leq \cos \theta \leq 1\]

in 2-D and 3-D, this is the cosine law.

6) if \( \theta = \frac{\pi}{2} \), (i.e., when \( x \cdot y = 0 \)), then \( x \) and \( y \) are orthogonal to each other.

Any another way to see this: Pythagoras' theorem:

\[\|(x+y)\|^2 = \|x\|^2 + \|y\|^2\]

\( x \perp y \) iff \( \|(x+y)\|^2 = \|x\|^2 + \|y\|^2 \)

\[\|(x+y)\|^2 = (x+y) \cdot (x+y) = \|x\|^2 + 2x \cdot y + \|y\|^2\]
holds iff \( x - y = 0 \).

1) Cauchy-Schwarz inequality leads to
\[
\| x + y \| \leq \| x \| + \| y \|
\]
(triangle inequality)

4), 5), 7) proof will be posted online.

8) a) in Matlab
\[
x = [1; 2; 3]
y = [4; 5; 6].
\]
\[
\text{dot}(x, y)
\]
\[
x' + y
\]
5) two subspaces \( V \) and \( W \) are said to be orthogonal to each other if every vector in \( V \) is perpendicular to every vector in \( W \).
written \( V \perp W \)
9) the orthogonal complement \( V^\perp \) of \( V \) is the subspace containing all vectors \( \perp \) to all vectors in \( V \). This subspace is denoted \( V^\perp \).

(a) \( W = V^\perp \)

(b) \( W \neq V^\perp \) because it doesn't contain all vectors \( \perp \) to \( V \).