last time:

* let $S$ be a subset of a vector space $V$ (e.g., $\mathbb{R}^4$)
  if $u$, and $v$ are elements of $S$, then $S$ is a subspace iff
  
  $c_1 u + c_2 v$

  is also an element of $S$.

* $\text{span} \{ u, \ldots, v_n \}$ is the set of all possible
  l.c.'s of $\{ u, \ldots, v_n \}$; it is a subspace.

Ex what geometrically, what is $\text{span} \{ (-1, 1) \}$?
$\mathbb{R}^2$

Ex what is $\text{span} \{ (\frac{1}{2}), (\frac{1}{2}) \}$?
the plane $x_3 = 0$

Ex is the vector $\left( \frac{1}{3} \right)$ contained in
$\text{span} \{ (\frac{1}{2}), (\frac{1}{2}) \}$?
\[ \Rightarrow \text{does there exist a solution to the system} \]
\[
\begin{pmatrix}
1 & 1 & c_1 \\
1 & 2 & c_2 \\
1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} \\
\frac{3}{2}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{3}
\end{pmatrix}
\]

\[ \Rightarrow \text{no solution} \]

\[ \Rightarrow (\frac{1}{2}, \frac{3}{2}) \text{ is not in } \text{span} \{ (\frac{1}{2}), (\frac{1}{2}) \} \]

\[ \Leftrightarrow 13 (\frac{2}{3}) \text{ in } \text{span} \{ (\frac{1}{2}), (\frac{1}{2}) \} \]
yes:
\[
\begin{pmatrix}
1 & 1 & 2 \\
1 & 2 & 3 \\
1 & 2 & 3
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]
\(rref(A)\)

row 2:  \(c_2 = 1\)
row 1:  \(c_1 = 1\)

\[
\begin{pmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix} = 1 \cdot \begin{pmatrix}
1 \\
1
\end{pmatrix} + 1 \cdot \begin{pmatrix}
1 \\
2
\end{pmatrix}
\]

\(\Rightarrow \begin{pmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix}\) is in \(span\{\begin{pmatrix}
1 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
2
\end{pmatrix}\}\).

\(\Rightarrow \begin{pmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix}\) is in \(span\{\begin{pmatrix}
1 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
2
\end{pmatrix}\}\) and \(span\{\begin{pmatrix}
1 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
2
\end{pmatrix}, \begin{pmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix}\}\)

\(\sim S_2\)

different? No.

any \(a.l.c.\) of vectors in \(S_2\) can be written as a \(a.l.c.\) of vectors in \(S_1\):

\[v = c_1 \begin{pmatrix}
1 \\
1
\end{pmatrix} + c_2 \begin{pmatrix}
1 \\
2
\end{pmatrix} + c_3 \begin{pmatrix}
\frac{2}{3} \\
\frac{2}{3}
\end{pmatrix}\]  \(\in \text{span}\{\begin{pmatrix}
1 \\
1
\end{pmatrix}, \begin{pmatrix}
1 \\
2
\end{pmatrix}\}\)

c are in \(S_1\)
so in a sense, \((\frac{2}{3})\) is "redundant".

What is the minimum \# of vectors we need to span a space?

**Basis**

A collection of vectors \(\{v_1, \ldots, v_n\}\) in a subspace \(V\) is called a basis for \(V\) if

1) \(\text{Span} \{v_1, \ldots, v_n\} = V\) so any vector in \(V\) can be expressed as a linear combination of \(v_1, \ldots, v_n\).

2) \(\{v_1, \ldots, v_n\}\) is a linearly independent set \(\rightarrow\) the l.c. is unique.

Why is it unique?

Assume not unique.

\[c \in V\]

\[v = c_1 v_1 + \ldots + c_n v_n\]

\[- v = d_1 v_1 + \ldots + d_n v_n\]

\[0 = (c_1 - d_1) v_1 + \ldots + (c_n - d_n) v_n\]

Since \(v_1, \ldots, v_n\) are l.i., then \(a_1, \ldots, a_n\) must all be 0.
$c_1 = d_1, \ldots, c_n = d_n \quad \forall \mathbf{c} \in \mathbb{R}^n$

i.e., if $A = (v_1, v_2, v_n)$ the solution to $A \mathbf{x} = \mathbf{y}$ is unique (if the solution exists)

$\mathbf{y} \in \mathbb{R}$

A basis is never unique

$\{ (1), (0) \}$ spans $\mathbb{R}^2$ is a basis for $\mathbb{R}^2$

$\{ (1), (-1) \}$ is also a basis

$\Rightarrow v_1, v_2, v_3 \in \mathbb{R}^n, n > 3$

Example: suppose $\{ v_1, v_2, v_3 \}$ forms the basis of a subspace. Let $P$ be a $m \times n$ invertible matrix. Let $V = (v_1, v_2, v_3)$. The columns of $VP$ forms a basis for the same subspace. What are the columns of $VP$?

$P = (p_1, p_2, p_3)$

$VP = (v_{p_1}, v_{p_2}, v_{p_3})$

different e.c.'s of $v_1, v_2, v_3$
to show that the col's of VP form a basis for the same subspace, need to show

1) they are l.i.
2) they span the same space

1) need to show \( VPx = 0 \) has only the trivial soln \( y \)

find soln to \( Vy = 0 \)

\[ y = P0 \] is the only soln because \( V \) has \( l.i. \) col's.

\[ y = Px = 0 \]

\( x = 0 \) is the only soln because \( P \) is invertible

\( \Rightarrow VP \) has l.i. col's
2) any vector in \( \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) can be written as

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3
\]

\[
= \mathbf{V} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
\]

so we need to solve

\[
\mathbf{V} \begin{pmatrix} \mathbf{P} & \mathbf{x} \\ \mathbf{y} & \mathbf{z} \end{pmatrix} = \mathbf{V} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
\]

\[
\mathbf{P} \mathbf{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
\]

is a solution.

\[
\mathbf{x} = \mathbf{P}^{-1} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}
\]

\Rightarrow any vector in \( \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) can be written as a.l.c. of the columns of \( \mathbf{V} \mathbf{P} \)

(and vice versa)

\Rightarrow they span the same space.
A basis for the vector space $P_3$ of polynomials of degree at most 3 is $\{1, x, x^2, x^3\}$. i.e., any element of $P_3$ can be written as

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 + c_4 \cdot x^3$$

To show that $\{1, x, x^2, x^3\}$ is a linearly independent set, we need to show

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 + c_4 \cdot x^3 = 0$$

has only the trivial solution for $c_1, c_2, c_3, c_4$.

Set $x = 0$:

$$c_1 = 0$$

Diff:

$$c_2 + 2c_3 \cdot x + 3c_4 \cdot x^2 = 0$$

$x = 0$:

$$c_2 = 0$$

Diff:

$$2c_3 + 6c_4 \cdot x = 0$$

$x = 0$:

$$c_3 = 0$$
\[ \text{diff} : \quad 6c_4 = 0 \]
\[ c_4 = 0 \]

just as before, this basis is not unique.
\[ \{1, (x-2), (x-2)^2, (x-2)^3\} \]

is also a basis (recall cubic spline interpolation).

Notice that these two bases have the same number of elements. In general, if two bases span the same space, they have the same # of elements. This number is called the dimension of the subspace.

The subspace vector subspace of all polynomials of degree at most \( n \) in \( \mathbb{R} \) has dimension \( n+1 \).

Show this: \( x \in \mathbb{R}^k \)

Suppose \( \{ v_1, \ldots, v_n \} \) and \( \{ w_1, \ldots, w_m \} \)

\( n < k \), \( m < k \).

These are two bases for a subspace \( S \).
we need to show $n = m$

since $\{v_1, \ldots, v_n\}$ span $S$, we can write any $w_j$ as

$$w_j = \sum_{i} a_{ij} v_i = a_{ij} v_1 + a_{2j} v_2 + \ldots$$

i.e.

$$
\begin{pmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_m
\end{pmatrix} = 
\begin{pmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_m
\end{pmatrix}
\cdot
\begin{pmatrix}
  a_{11} & \cdots & a_{1m} \\
  \vdots & \ddots & \vdots \\
  a_{n1} & \cdots & a_{nm}
\end{pmatrix}
$$

$$W = VA$$

now try to solve

$$Wx = 0$$

what happens if $m > n$?

$$V[A][x] = 0$$

$A \cdot x = 0$ 

guaranteed non-trivial soln
if \( m > n \), \( VAx = 0 \) has a nontrivial solution since \( A x = 0 \) has a \( u \) solution then 

\[ W x = 0 \]

has a nontrivial solution. But this is impossible since \( \text{cl}(W) \) are l.i. \( \Rightarrow m \) cannot be greater than \( n \)

reverse this argument:

\[ V = WB \]

\[ m \times n \]

same procedure:

\( m \) cannot be greater than \( m \)

\[ \Rightarrow n = m \]

Remarks:

1) any set of \( n \) l.i. vectors \( \{v_1, \ldots, v_n\} \) contained in a \( k \)-dimensional subspace \( S \) is automatically a basis for that subspace.

To see this, suppose \( \{w_1, \ldots, w_n\} \) is also a basis for \( S \).
let $V = (v_1, \ldots, v_n)$, $W = (w_1, \ldots, w_n)$

then

$V' = W[A] \quad$ (as before)

If try to solv $Vx = 0$

matrix $[WA]x = 0$

A must be invertible since $Vx = 0$

has only the trivial solution (i.e. col's are l.i.). If A was not invertible, i.e.

if $Ay = 0$ for $y \neq 0$, then

$W[Ay] = 0$

so WA would have a non-trivial nullspace.

and from before, we know that col's of WA and W must span the same space when $A$ is invertible $\Rightarrow$ col's of $V$ form basis of the same subspace.
2) any vector in a \( k \)-dimensional subspace spanned by \( \{ v_1, \ldots, v_k \} \) can be identified by \( k \) numbers, i.e., the coefficient \( c_j \) of

\[ c_1 v_1 + c_2 v_2 + \ldots + c_k v_k \]

If basis is \( \{ 1, x, x^2, x^3 \} \), then \( 1 + x^2 + 2x^3 \) can be represented as

\[
\begin{pmatrix}
1 \\
0 \\
1 \\
2
\end{pmatrix}
\]

We can add these vectors as usual:

\[
\begin{pmatrix}
1 \\
0 \\
1 \\
2
\end{pmatrix} + \begin{pmatrix}
0 \\
-1 \\
1 \\
4
\end{pmatrix} = \begin{pmatrix}
1 \\
-1 \\
5 \\
3
\end{pmatrix}
\]

\[ \Rightarrow 1 - x + 5x^2 + 3x^3. \]

If basis was \( \{ 1, (x-1), (x-1)^2, (x-1)^3 \} \),

\[
\begin{pmatrix}
1 \\
0 \\
1 \\
2
\end{pmatrix} \rightarrow 1 + (x-1) + (x-1)^2 + 2(x-1)^3
\]

So it depends on what the basis is.
The four fundamental spaces of a matrix.

\[
\begin{bmatrix}
  \vdots \\
  \vdots
\end{bmatrix}
\begin{bmatrix}
  \vdots \\
  \vdots
\end{bmatrix} = 0
\]

Let \( A \) be an \( n \times m \) matrix, and let \( x \in \mathbb{R}^m \).

The nullspace (\( N(A) \)) or kernel (\( \ker(A) \)) of \( A \) is the set of all vectors \( x \) s.t.

\[ A x = 0 \]

\( x \in N(A) \) iff \( A x = 0 \).

If \( \text{col's of } A \) are l.i., then \( N(A) = \{ 0 \} \).

\( N(A) \) is a subspace of \( \mathbb{R}^n \).

If \( x_1 \in N(A) \), \( x_2 \in N(A) \)

\[ A \left( c_1 x_1 + c_2 x_2 \right) = c_1 A x_1 + c_2 A x_2 \]

\[ = c_1 0 + c_2 0 \]

The column space (\( C(A) \)) or range (\( R(A) \)) is the set of all possible l.c.'s of the columns of \( A \).
If \( A = (a_1, \ldots, a_n) \),

the \( R(A) = \text{span}\{a_1, \ldots, a_n\} \).

the other two spaces are \( N(A^T), R(A^T) \).