Assignment 7

(1) Consider the sequence of numbers for which each number is the average of the previous two numbers, i.e.,

\[ G_{k+2} = \frac{1}{2} (G_{k+1} + G_k) ; \quad G_0 = 0, \quad G_1 = \frac{1}{2}. \]  

Find a formula for \( G_k \), and compute its limit as \( k \to \infty \).

(2) We looked at the boundary value problem \( u'' = 0 \) with either Neumann \( (u'(0) = u'(1) = 0) \) or Dirichlet \( (u(0) = u(1) = 0) \) boundary conditions. Recall that the solution \( u(x) \) gives the steady state heat distribution of a rod with no external heating source. We found that with Neumann boundary conditions (no heat gets in or out), the solution is non-unique; \( u(x) = c \) is a solution for any constant \( c \). In the Dirichlet case (heat can leave through the ends), \( u(x) \equiv 0 \) is the only solution. In this question, we will see how these two cases differ physically. We will also see the process by which the steady state distribution is reached from a given initial condition. The model will be based on a system of random walkers.

Consider \( N \) spaces arranged side-by-side (picture a single row of parking spaces). Label these locations 1, 2, \ldots, \( N \). Each space has a certain number of people standing it in (either zero or nonzero). Every second, each person currently in spaces 2, \ldots, \( N-1 \) will leave the space s/he is standing in, and go one space to the left or to the right with equal probability. Denote \( u_j(t) \) the number of people in space \( j \) at time \( t \) seconds, and let

\[
\mathbf{u}(t) = \begin{pmatrix}
u_1^{(t)} \\
u_2^{(t)} \\
\vdots \\
u_N^{(t)}
\end{pmatrix}.
\]

In the parts below, we will consider two different rules for what happens at spaces 1 and \( N \).

(a) If a person is at space 1, s/he will stay at location 1 with probability 1/2 and move to space 2 with probability 1/2. If a person is at space \( N \), s/he will stay at space \( N \) with probability 1/2 and move to space \( N-1 \) with probability 1/2. With this rule, for \( N = 4 \), give the 4 \times 4 stochastic matrix such that

\[
\mathbf{u}(t+1) = A \mathbf{u}(t).
\]

(b) Let \( N = 100 \). Suppose that there are originally (at \( t = 0 \)) 500 people in space 50, 500 people in space 51, and none in all the other spaces. On the same graph, use Matlab to plot the number of people in each space at times \( t = 0, t = 10, t = 50, \) and \( t = 2000 \). This can be done simply with the command \texttt{plot(ut)} , where \texttt{ut} is the Matlab variable representing the vector in Eqn. (2). Use \texttt{hold on} so that Matlab does not erase the previous plot. Submit the plot. Do not submit code.

(c) As \( t \to \infty \), how many people are in each space? If instead at the beginning there were 1000 people in space 50, 1000 people in space 51, and none in all the other spaces, how many people will be in each space as \( t \to \infty \)? The different answers due to the different initial conditions is the source of the non-uniqueness in the Neumann case. You may use Matlab to assist in obtaining your answers. Do not submit code.

(d) Repeat Part (2a), except with the following rule for spaces 1 and \( N \): everyone currently in spaces 1 and \( N \) will exit the system at the next second (i.e., everyone currently in spaces 1 and \( N \) simply leave and go home). Note that the matrix \( A \) in this case will not be a stochastic matrix.

(e) Repeat Part (2b) for times \( t = 0, t = 10, t = 50, \) and \( t = 3000 \). Do the graphs resemble a familiar probability distribution?

(f) For this second rule, how many people will be in each space as \( t \to \infty \)? Does it depend on initial conditions? This scenario corresponds to the Dirichlet boundary conditions.
(3) Use one of the two procedures from class to compute the singular value decomposition of the matrix

\[ A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]  

(4) Show all your work. Do not simply write down the `svd()` answer from Matlab. Remember that you can always verify your answer.