Assignment #6 solutions

1. \( \lambda_1 = 2, \quad \nu_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \)

   \( \lambda_2 = 3, \quad \nu_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)

   \( \lambda_3 = 3, \quad \nu_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \)

   Construct

   \[ D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad S = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \]

   Then

   \[ A = SDS^{-1} \] will have the above eigenpairs.

MATLAB gives

\[ A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{pmatrix} \]
have evo's 0, 0, 0.

To be diagonalizable, $A = 0I$ must have a nullspace of dimension 3 (geometric multiplicity must equal algebraic multiplicity, for diagonalizability).

However, clearly $\text{rank}(A) = 1$, so $\dim \text{Nul}(A) = 2 \Rightarrow$ Not diagonalizable.

(3) $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is diagonalizable (it already is diagonal), but not invertible.

$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.
indeed, the two ev's of C are \( \frac{1}{4} \)
but \( \dim \text{Nul} (A-I) = 1 \)
\[ \uparrow \]
\[ \text{geom. mult.} = 2 \]
\[ \text{alg. mult.} = 1 \]

4.b) for part (a), see below. should get
\[ \lambda \approx -28.0286 \]

since we are trying to estimate the
largest eigenvalue, we can track this
estimate and terminate the algorithm
when the estimates start to change by
very little. So suppose that \( \lambda_j \) is
the \( j \)th estimate:

\[ \lambda_j = X_j^T A X_j \]

then we stop when the following condition
is satisfied:

\[ |\lambda_{j+1} - \lambda_j| < 3 \]
where $\varepsilon$ is some small number.
The smaller the $\varepsilon$, the more accurate the final estimate, but the longer the computations will take.

Using

$$\| z_{j+1} - z_j \| < \varepsilon$$

is not as good, since if the largest eigenvector is negative, $z_j$ will simply oscillate between $\nu$ and $-\nu$, where $\nu$ is the normalized eigenvector corresponding to the largest eigenvalue. To get around that, we can do this instead:

$$\| z_{j+2} - z_j \| < \varepsilon$$
$\lambda \approx -28.0286$
5) For the code, see below.

You should obtain that $l_1 = -1$

6) a) if $A = B^*B$, and $\lambda$ is an eigenvector of $A$, then

$$Ax = \lambda x$$

$$x^*Ax = \lambda x^*x$$

$$x^*B^*Bx = \lambda x^*x$$

$$\text{(Bx)}^*Bx = \lambda x^*x$$

$$\|Bx\|^2 = \lambda \|x\|^2$$

$$\|Bx\|^2 = \lambda \|x\|^2$$

$$x = \frac{\|Bx\|^2}{\|x\|^2} > 0 \quad \text{and} \quad \lambda \text{ real}$$
# 5 code

clear all
close all

load matrixforq5.mat

N = length(A);
x = rand(N,1);
s = 0;
B = A - s*eye(N);

iterations = 30;

for k = 1:1:iterations;
    y = B \ x;
    x = y/norm(y);
end

lambda_max = x'*A*x;
b) if $A$ is an $n \times n$ with all real eigenvalues and $n$ mutually orthogonal eigenvectors, then

$$A = VDV^*$$

where

$$AV_j = \lambda_j V_j, \quad \lambda_j \text{ real}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix} \quad \leftarrow D \text{ real}$$

$$V = \begin{pmatrix} V_1 & V_2 & \cdots & V_n \end{pmatrix}$$

$$\|V_j\| = 1, \quad j = 1, \ldots, n$$

$$V_j^*V_k = 0 \quad \text{if } j \neq k.$$ 

$$A^* = (V^*D^*V^*)^* = (V^*)^*D^*V^*$$

$$= VDV^* = A$$

$D$ is real

$$A^* = A \Rightarrow A \text{ is Hermitian}$$
c) i) if \( A \) is invertible, then
\[
\det(A - \lambda I) \neq 0
\]
\[\Rightarrow \lambda \text{ is not an eigenvector of } A\]

ii) \( A \mathbf{x} = \lambda \mathbf{x} \) \( A^{-1} \) exists.
\[
\mathbf{x} = \lambda A^{-1} \mathbf{x}
\]
\[
\frac{1}{\lambda} \mathbf{x} = A^{-1} \mathbf{x}
\]
\[\Rightarrow \frac{1}{\lambda} \text{ is an eigenvector of } A^{-1} \text{ with}
\]
the eigenvector \( \mathbf{x} \), where \( A \mathbf{x} = \lambda \mathbf{x} \)

iii) \( \det(\lambda I - A) = p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n) \)
\[p\left(\frac{1}{\lambda}\right) = \left(\frac{1}{\lambda} - \lambda_1\right)\left(\frac{1}{\lambda} - \lambda_2\right) \ldots \left(\frac{1}{\lambda} - \lambda_n\right)\]
\[p\left(\frac{1}{\lambda}\right) = 0 \text{ when } \lambda = \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}
\]
the roots of \( p\left(\frac{1}{\lambda}\right) \)
the roots correspond to the eigenvectors of \( A^{-1} \).
iv) a polynomial is uniquely determined by its roots and the coefficient on the term with the highest power.

\[ \det (\lambda I - A^t) \] has roots at \( \lambda = \frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n} \), as does \( p(\frac{1}{\lambda}) \). So we just need to match the powers of \( p(\frac{1}{\lambda}) \) to that of \( \det (\lambda I - A^{-1}) \), along with the coeff. of the highest power.

Notice that, the coefficient of highest power is 1

\[ \det (\lambda I - A^{-1}) = \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_0 \]

while

\[ p(\frac{1}{\lambda}) = \frac{1}{\lambda^n} + a_{n-1} \frac{1}{\lambda^{n-1}} + \ldots + (-1)^n (\lambda_1 \lambda_2 \ldots \lambda_n) \]

so

\[ \lambda^n p(\frac{1}{\lambda}) = (-1)^n (\lambda_1 \ldots \lambda_n) \lambda^n + a_1 \lambda^{n-1} + \ldots + 1 \]

Coefficient of highest power is \( (-1)^n (\lambda_1 \ldots \lambda_n) = (-1)^n \det A \)
\[ \frac{\lambda^n}{(-1)^n \det A} p\left(\frac{1}{\lambda}\right) \] will have coefficient 1 on the \( \lambda^n \) term, and have the same roots as \( \det (\lambda I - A^{-1}) \).

\[ \Rightarrow \det (\lambda I - A^{-1}) = \frac{\lambda^n}{(-1)^n \det A} p\left(\frac{1}{\lambda}\right) \]

where

\[ p(\lambda) = \det (\lambda I - A). \]