1) if $A \cdot x = 0$, then $A^T A \cdot x = A^T 0 = 0$

b) if $A^T A \cdot y = 0$, then $A \cdot y \in N(A^T) = R(A)^\perp$

so $A \cdot y \perp$ to every vector in $R(A)$.

$\Rightarrow (A \cdot y)^T A \cdot z = 0$ for any $z$

take $z = y$. Then

$$(A \cdot y)^T A \cdot y = 0$$

$\|A \cdot y\|^2 = 0 \Rightarrow A \cdot y = 0$

i.e., $A \cdot y$ must be $\perp$ to itself, so it must be the $0$ vector.

c) from a) we show that $N(A)$ is a subspace of $N(A^T A)$. from b) we show that $N(A^T A)$ is a subspace of $N(A)$. 
we can conclude that \( N(A) \) and \( N(A^T A) \) are the same space.

a) If \( A x = 0 \) has only the trivial solution, then since \( A^T A \) has the same nullspace as \( A \), we must have that \( A^T A x = 0 \) has only the trivial solution. \( \Rightarrow A^T A \) is invertible.

If \( A^T A \) is invertible, \( A^T A x = 0 \) has only the trivial solution. Then \( A x = 0 \) must have only the trivial solution.

2) \( N(A) = N(A^T A) \)

But \( N(A) = R(A^T)^\perp \)

and \( N(A^T A) = R((A^T A)^T)^\perp = R(A^T A)^\perp \)

\( \Rightarrow R(A^T)^\perp = R(A^T A)^\perp \)
since \((V^\perp)^\perp = V\), take “perp” of both sides, we have then
\[ R(ATA) = R(A^2A) \]

3) the columns of \(A\) must be linear combinations of the columns of \(B\), since the columns in both matrices form a basis of the same subspace.

\[ \Rightarrow A = BC \]

where \(C\) is square and invertible, then
\[ A(A^2A)^{-1}A^T = BC \begin{pmatrix} C^T B^T B C & \end{pmatrix}^{-1} C^T B^T \]

\[ = BC C^{-1} \begin{pmatrix} B^T B \end{pmatrix}^{-1} \begin{pmatrix} C^T \end{pmatrix}^{-1} C^T B^T \]

\[ = B \begin{pmatrix} B^T B \end{pmatrix}^{-1} B^T \]
the projection matrix therefore does not depend on the choice of basis.

(4) Form a matrix \( B \) whose columns are a basis for \( \text{Col}(A) \). Pick the pivot columns of \( A \):

\[
B = \begin{pmatrix}
1 & 2 \\
5 & 6 \\
-1 & 2 \\
3 & 2
\end{pmatrix}
\]

Matlab:

\[
B = \begin{bmatrix}
1 & 2 \\
5 & 6 \\
-1 & 2 \\
3 & 2
\end{bmatrix}
\]

\[
P = B \times \text{inv}(B' \times B) \times B'
\]

(1 or transpose both okay)

\[
P = \begin{pmatrix}
1/9 & 2/9 & 2/9 & 0 \\
2/9 & 7/9 & 1/9 & 3/9 \\
2/9 & 1/9 & 7/9 & -3/9 \\
\end{pmatrix}
\]
5) 

a) $y - x \in N(P)$ but $N(P) = \mathbb{R}(Q)$

where $Q = I - P$ (remember that $P$ is a projection matrix). So $y - x \in \mathbb{R}(Q)$

then $y - x = Qz$ for any vector $z$

\[
\begin{align*}
  y &= x + Qz \\
  y &= x + Qz
\end{align*}
\]

b) $Py$ is the closest point. But

$Py = Px + PQz$

where $PQ = P(I - P) = P - P^2 = P - P = 0$.

so even though there are an infinity of possibilities for $y$, there is only one $Py$, i.e., one closest point in $\mathbb{R}(P)$ to $x$. 
\[ a) \quad p_i = \beta_0 + \beta_1 \log w_i \]
\[ p_n = \beta_0 + \beta_1 \log w_n \]

\[
\begin{bmatrix}
\log w_1 \\
\vdots \\
\log w_n
\end{bmatrix}
\begin{bmatrix}
\beta_0 \\
\beta_1
\end{bmatrix}
= 
\begin{bmatrix}
p_1 \\
p_n
\end{bmatrix}
\]

\[ A^T A \mathbf{c} = A^T \mathbf{p} \]

\[ \mathbf{c} = (A^T A)^{-1} A^T \mathbf{p} \]

**Matlab** ...

\[
\begin{bmatrix}
\beta_0 = 19.5539 \\
\beta_1 = 19.6045
\end{bmatrix}
\]

\[ \beta_0 + \beta_1 \log 85 = 106.6501 \]