1. Find the equation of the tangent line to the graph of

\( y = \sin(x)^2 \)

at the point where \( x = \frac{\pi}{3} \).

[20 marks]

\[ \frac{dy}{dx} = 2 \sin(x) \cos(x) \quad 5 \text{ marks} \]

At \( x = \frac{\pi}{3} \)

\[ \frac{dy}{dx} = 2 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad 5 \text{ marks} \]

Solve for \( y \) when \( x = \frac{\pi}{3} \)

\[ y = \sin\left(\frac{\pi}{3}\right)^2 = \frac{3}{4} \quad 5 \text{ marks} \]

\[ y = \frac{\sqrt{3}}{2} x + b \]

Solve for \( b \)

\[ \frac{3}{4} = \frac{\sqrt{3}}{2} \pi + b \]

\[ b = \frac{3}{4} - \frac{\sqrt{3}}{2} \pi \]

\[ y = \frac{\sqrt{3}}{2} x + \frac{3}{4} - \frac{\sqrt{3}}{2} \pi \quad 5 \text{ marks} \]
2. You are flying a kite. At a certain time, the kite is 30 m high, 40 m horizontally away from you and moving horizontally away from you at a rate of 10 m/min. Assume that the string lies on a straight line between you and the kite at all times.

[10 marks]

a) How fast are you letting out string at that time?

Let \( r \) be the distance between you and the kite.

By the Pythagorean Theorem.

\[
\frac{2}{r} = \frac{x}{r} + \frac{y}{r} \quad 2 \text{ marks}
\]

Solve for \( r \). We find \( r = 50 \).

Then by differentiating implicitly with respect to time

\[
2r \frac{dr}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \quad 5 \text{ marks}
\]

By substituting in values, we have:

\[
\frac{dy}{dt} = 0 \quad \frac{dx}{dt} = 10 \quad y = 30 \quad x = 40 \quad r = 50
\]

\[
250 \frac{dr}{dt} = 2 \times 40 \times 10 + 2 \times 30 \times 0
\]

\[
\frac{dr}{dt} = 8 \text{ ft/min} \quad 3 \text{ marks (1 mark for units)}
\]
b) How fast is the angle between the string and the ground changing at that time?

\[ \tan(\theta) = \frac{y}{x} \quad 3 \text{ marks} \]

Differentiating with respect to time, we have

\[ \sec^2(\theta) \frac{d\theta}{dt} = \frac{x \frac{dy}{dt} - y \frac{dx}{dt}}{x^2} \quad 4 \text{ marks} \]

Substituting in

\[ \frac{dy}{dt} = 0, \quad \frac{dx}{dt} = 10, \quad y = 30, \quad x = 40, \quad \sec^2(\theta) = \frac{25}{16} \]

\[ \frac{d\theta}{dt} = \frac{3}{25} \text{ radian/min} \quad 3 \text{ marks (1 mark for units)} \]
a) Use any method to find the tangent line to the circle
\[ x^2 + y^2 - 12x - 6y + 25 = 0 \]

at the point (2,1). Show your work.

[10 marks]

By implicit differentiation:

\[
2x + 2y \frac{dy}{dx} - 12 - 6 \frac{dy}{dx} = 0
\]

5 marks

Substituting in \(x = 2\), \(y = 1\)

\[
\frac{dy}{dx} = -2
\]

2 marks

Substituting in \(x = 2\), \(y = 1\)

\[
v = -2x + b
\]

1 = -4 + b

b = 5

v = -2x + 5

3 marks
Alternate solution: by using Geometry

Any line segment from the center of a circle to a point P is perpendicular to the tangent line at the point P.

\[
\text{Slope of } CP = \frac{\Delta y}{\Delta x} = \frac{3 - 1}{6 - 2} = \frac{1}{2}
\]

The slope of a line that is perpendicular to the original line is equal to the negative-inverse slope of the original line.

\[
\text{Slope of tangent line} = -\frac{1}{2} = -0.5
\]

7 marks

b) Using implicit differentiation find the slope of the tangent line to the circle

[10 marks]

\[x^2 + y^2 + 2x + y - 10 = 0\]

At the point (2, 1).

Implicitly differentiating,

\[2x + 3 \frac{dy}{dx} + 2 = 0\]

7 marks

Substituting in with x=2 y=1

\[
\frac{dy}{dx} = -2
\]

3 marks
4. Find the point on the graph of \( y = \sqrt{x} \) closest to the point \((4,0)\).

[20 marks]

Let \( B(x, \sqrt{x}) \) be an arbitrary point on the curve \( y = \sqrt{x} \).

Let \( r \) be the distance between a point on the curve and the point \((4,0)\)

\[
r = \sqrt{(4-x)^2 + x^2} = \sqrt{16 - 7x + x^2}
\]

10 marks

\[
\frac{dr}{dx} = \frac{-7 + 2x}{2 \sqrt{16 - 7x + x^2}}
\]

5 marks

Solving \( \frac{dr}{dx} = 0 \)

\[
\frac{dr}{dx} = \frac{-7 + 2x}{2 \sqrt{16 - 7x + x^2}} = 0
\]

\[-7 + 2x = 0
\]

\[
x = \frac{7}{2}
\]

5 marks

The point on the curve \( y = \sqrt{x} \) closest to the point \((4,0)\) is \( \left( \frac{7}{2}, \sqrt{\frac{7}{2}} \right) \).
5. It is desired to use Newton’s Method to find the value of x at which the function

\[ y = 2e^x + x^2 - 4x + 1 \]

has a horizontal tangent line. If you start with \( x_0 = 0 \) find \( x_1 \). Note that you are only required to find \( x_1 \).

[20 marks]

Note: We are trying to find when the slope is horizontal (equal to 0) so we are trying to solve when \( dy/dx = 0 \) \( NOT \) when \( y = 0 \), therefore, we must apply Newton’s Method to \( dy/dx \) instead of \( y \).

\[ \frac{dy}{dx} = 2e^x + 2x - 4 \]

Let \( h(x) \) be \( dy/dx \) for convenience.

\[ h(x) = 2e^x + 2x - 4 \] 5 marks

\[ h'(x) = 2e^x + 2 \] 5 marks

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \] 5 marks

\[ x_1 = x_0 - \frac{h(x_0)}{h'(x_0)} \]

\[ x_1 = 0 - \frac{2e^0 + 2 - 4}{2e^0 + 2} = \frac{1}{2} \] 5 marks