
For each kid, call X, we know that all the kids of the opposite group have solved at least one problem in common with X. Since X solves at most six problems and there are 21 kids in the opposite group, there cannot be more than 10 kids in the opposite group that solved only two (or less) problems in common with X. Thus, the rest, at least 11, had solved at least three problems in common with X. (This is a simple application of the pigeonhole principle where the problems solved by X are the holes and the opposite group are the pigeons, and the “pigeon” enters a hole/problem which he solved in common with X).

Now, let’s focus on one group, let’s say the boys. The boys have solved at most $21 \times 6$ problems. There could be repetitions here since boys could have solved the same problem. We define A as the set with all the problems that a boy had solved with repetitions, i.e if a problem is solved by more than one boy it appears as many times as the number of boys who have solved it. Clearly, A has size $21 \times 6$.

Now, let’s mark each problem in A that has been solved by at least three boys and a girl. Each problem can have multiple marks, the same as the number of girls that has solved it. We want to prove that there is one problem with 3 marks. There has to be a total of $21 \times 11$ marks since there are at least 11 marks for each girl (by the discussion above). Thus there are more marks than problems in A ($21 \times 11 > 21 \times 6$). If a problem is marked three times, we are done. Let’s assume that all problems are marked at most twice. Thus, there are at most $\frac{21 \times 11}{2}$ problems marked. In particular, there are more than $21 \times 5$ marked problems in A ($21 \times 5 \times 2 < 21 \times 11$). Thus, there is a boy such that he solved six problems (otherwise we have fewer problems than marks which will contradict the first paragraph) and his problems had been marked. But then by our discussion in the first paragraph, we know there must a problem that this boy has solved which has been solved by at least three girls. Therefore it must be true that there is a problem such that it has been solved by at least three people of each gender.

2. (a) Use induction on $m$. Firstly, for $m = 1$ the formula is $u_{n+1} = u_0 u_n + u_1 u_{n+1} = u_{n+1}$, so it is true for all $n$. 

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Now suppose that the formula holds for fixed \( m \) and all \( n \). Then for \( m + 1 \) we obtain

\[
\begin{align*}
  u_{n+(m+1)} &= u_{(n+1)+m} \\
  &= u_{m-1}u_{n+1} + u_m u_{n+2} \quad \text{by hypothesis of induction} \\
  &= u_{m-1}u_{n+1} + u_m(u_n + u_{n+1}) \quad \text{by definition of Fibonacci} \\
  &= u_{m-1}u_{n+1} + u_m u_{n+1} + u_m u_n \\
  &= (u_{m-1} + u_m)u_{n+1} + u_m u_n \\
  &= u_{m+1}u_{n+1} + u_m u_n \quad \text{by definition of Fibonacci}
\end{align*}
\]

which is the formula we want for \( m + 1 \). This concludes the induction. Hence if \( x \) divides both \( u_n \) and \( u_m \) then it divides any linear combination of both, in particular, \( x \) divides \( u_{n+m} = u_{m-1}u_n + u_m u_{n+1} \).

For the second part, we apply induction again, now on \( k \). For \( k = 1 \) the claim is \( u_n \) divides \( u_n \), which is true. Now suppose that \( u_n \) divides \( u_{kn} \). For \( k + 1 \) we obtain

\[
\begin{align*}
  u_{(k+1)n} &= u_{kn+n} \\
  &= u_{kn-1}u_n + u_{kn}u_{n+1} \quad \text{by the previous part}
\end{align*}
\]

By hypothesis of induction \( u_n \) divides \( u_{kn} \) (and clearly \( u_n \) divides \( u_n \)) hence \( u_n \) divides \( u_{(k+1)n} \).

(b) Now, if \( n \) is not prime, then \( n = p \times q \) where at least one of \( p \) or \( q \) is greater than 2 (except if \( n = 4 \)), assume without lost of generality that \( q \geq 3 \). Then, by the previous result we have that \( u_n \) is divisible by \( u_q \geq 2 \), so \( u_n \) cannot be prime if \( n \) is not prime and greater than 4. (The case \( n = 4 \) is special and \( u_4 = 3 \) is indeed prime)

Now we check the first 20 Fibonacci numbers, since we are looking for primes, we need to check only when \( n \) is prime or 4. We already now that \( u_2 = 1 \) is not prime whereas \( u_3 = 2 \) and \( u_4 = 3 \) are prime. Next \( u_5 = 5, u_7 = 13, u_{11} = 89, u_{13} = 233, u_{17} = 1597 \) which can be checked to be prime numbers, but \( u_{19} = 4181 = 37 \times 113 \) is not prime.
3. This is an application of the Chinese Reminder Theorem. The first condition can be neatly written as \( Z \equiv -1 \pmod{n} \) with \( n = 2, 3, 4, 5, 6 \) and the second condition is \( Z \equiv 0 \pmod{7} \).

Now, \( \text{m.c.m.}(2, 3, 4, 5, 6) = 60 \), so \( Z \) is of the form \( 60k - 1 \) and we want to solve \( 60k - 1 \equiv 0 \pmod{7} \), so

\[
\begin{align*}
60k &\equiv 1 \pmod{7} \\
4k &\equiv 1 \pmod{7} \\
8k &\equiv 2 \pmod{7} \\
k &\equiv 2 \pmod{7}
\end{align*}
\]

So \( k = 7m + 2 \). Then \( Z = 60(7m + 2) - 1 = 420m + 119 \). Therefore the smallest possible number of eggs in the basket is 119.

4. We have the system of equations

\[
\begin{align*}
x^2(1 + y^2 + y^4) &= 525, \\
x(1 + y + y^2) &= 35.
\end{align*}
\]

Multiply the first equation times \((y^2 - 1)\) and the second times \((y - 1)\), we get

\[
\begin{align*}
x^2(y^6 - 1) &= 525(y^2 - 1), \\
x(y^3 - 1) &= 35(y - 1).
\end{align*}
\]

Finally square the second and divide the two equations to get

\[
\frac{y^6 - 1}{(y^3 - 1)^2} = \frac{525(y^2 - 1)}{35^2(y - 1)^2}.
\]

Simplify using \( y^6 - 1 = (y^3 - 1)(y^3 + 1) \) and \( y^3 - 1 = (y - 1)(y^2 + y + 1) \), \( y^3 + 1 = (y + 1)(y^2 - y + 1) \), so

\[
y^2 - y + 1 = \frac{y^3 + 1}{y + 1} = \frac{525(y^3 - 1)}{35^2(y - 1)} = \frac{3(y^2 + y + 1)}{7}.
\]

We are left with a quadratic equation, \( 7y^2 - 7y + 1 = 3y^2 + 3y + 1 \), which is equivalent to \( 0 = 2y^2 - 5y + 2 = 2(y - \frac{1}{2})(y - \frac{1}{2}) \). Hence, the
solutions are $y = 2$ and $y = \frac{1}{2}$. Thus, we obtain $x = 5$ and $x = 20$, respectively.

Check that $(x, y) = (5, 2)$ and $(20, \frac{1}{2})$ satisfy the second equation. Therefore, these are the only solutions of the equations.

5. The description of the problem is depicted in the following picture, where $A$ is Alice’s eyes position, $B$ is the smokestack’s tip, $O$ is the center of the Earth and $P$ is the point of tangency of Alice’s line of sight with the Earth.

Now, the length of $AB$ can be computed using Pythagoras’ Theorem twice:

$$AP = \sqrt{AO^2 - OP^2} = \sqrt{6400.01^2 - 6400^2} = \sqrt{128.0001} = 11.31$$

$$BP = \sqrt{BO^2 - OP^2} = \sqrt{6400.04^2 - 6400^2} = \sqrt{512.0016} = 22.63$$

Therefore, $AB = AP + BP = 33.94 \text{ km}$. 