[20] 1. Fibonacci Nim has 2 players and a pile of sticks. The first player takes away at least 1 but not all the sticks from the pile. The second player can take away any number from 1 to twice the number the previous player took, then the first player does the same and so on. The strategy for the starting player is to start with a non-Fibonacci number and then to write as a sum of non-consecutive Fibonacci numbers and then discard the smallest.

[8](a) Find all the Fibonacci numbers less than 400.

\[
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377
\]

[6](b) Suppose you start with 42 sticks. What is your first move?

\[
42 = 34 + 8 \text{ as a sum of nonconsecutive Fibonacci numbers. Winning strategy is to remove smallest number in this sum}
\]

*First move: remove 8 sticks*

[6](c) Suppose you start with 220 sticks. What is your first move? Show your work.

\[
220 = 144 + 55 + 21
\]

First move is to remove 21 sticks, since it is the smallest number in the sum of nonconsecutive Fibonacci numbers that add to 220.
20. Prove that

13^{1/2} is irrational.

(b) 17 + 13^{1/2} is irrational, using the result from part (a).

5 a) Assume that \(\sqrt{13}\) is rational \(\left(\frac{m}{n}\right)\), where \(m\) and \(n\) are reduced.

\[\sqrt{13} = \frac{m}{n} \rightarrow \left(\sqrt{13}\right)^2 = \left(\frac{m}{n}\right)^2 \rightarrow 13 = \frac{m^2}{n^2} \rightarrow 13n^2 = m^2\]

\[n \mid m, \text{ thus } m^2 \text{ and } m \text{ are both numbers with } 13 \text{ as a factor}\]

Since \(m\) has 13 as a factor, it can be rewritten as \(13x\)

\[13n^2 = (13x)^2 = (13x)(13x) = 169x^2 \rightarrow \frac{13n^2}{13} = \frac{169x^2}{13} \rightarrow n^2 = 13x^2\]

\[n \mid 13x, \text{ thus } n^2 \text{ and } n \text{ are both numbers with } 13 \text{ as a factor}\]

This is a contradiction to our original assumption that \(\frac{m}{n}\) is reduced, since both \(m\) and \(n\) have 13 as a factor.

Thus our original assumption that \(\sqrt{13}\) is rational is also incorrect, so \(\sqrt{13}\) must be irrational.

5 b) Assume \(17 + \sqrt{13} = \frac{m}{n}\), where \(\frac{m}{n}\) is a fraction in lowest terms.

\[\sqrt{13} = \frac{m}{n} - 17 \Rightarrow \sqrt{13} = \frac{m - 17n}{n}\]

\[13m - 17n = n \Rightarrow m = \frac{17n}{n}\]

\[\frac{m}{n} \text{ is a fraction. Since we know that } \sqrt{13} \text{ is irrational,}\]

this is a contradiction.

Thus our original assumption that \(17 + \sqrt{13}\) is rational is incorrect, so it must be irrational.
3. A game show requires you to randomly pick 1 of 7 envelopes. The host has hidden a $100 bill in one envelope, and a $1 bill in each of the other 6. Once you've picked, the host is required to remove 2 of the $1 envelopes you didn't pick. You now have a choice of keeping your original envelope, or paying $2 to switch your choice to one of the 4 you didn't pick.

(a) If you choose to pay $2 and switch your choice, then what are your odds of losing money?

(b) By what percent (if any) does the odds of winning a $100 bill change by switching your choice?

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![diagram](image)

- First choice: \( \frac{1}{7} \) chance of winning $100, \( \frac{6}{7} \) chance of winning $1

- After host removes 1 envelopes, the original selection still has a \( \frac{1}{7} \) chance of winning, and the remaining envelopes still have a \( \frac{6}{6} \) chance of winning, now divided between 4 envelopes instead of 6.

- Each of the remaining envelopes has \( \frac{1}{4} \) chance of winning; \( \frac{6}{14} \) which of course means a \( \frac{10}{14} \) chance of losing (1 - \( \frac{6}{14} \) = \( \frac{8}{14} \)).

- If you switch envelopes, \( \frac{10}{14} \) chance of opening a $1 envelope, which means an \( \frac{10}{14} \) chance of losing money

(b) As seen above, original choice has \( \frac{1}{7} \) chance of winning. Also seen above: switch of choice results in \( \frac{3}{14} \) chance of winning

\[
\frac{1}{7} : \frac{2}{14} \quad \frac{2}{14} \Rightarrow \frac{3}{14} \quad \text{is an increase of} \quad \frac{1}{14} \quad \text{which is} \quad 50\% \quad \text{of} \quad \frac{2}{14}
\]

Thus the odds of winning a $100 bill increase 50% by switching choices.
[10](a) Solve the equation \( x = 1 + \frac{1}{x} \) for \( x \). Hint: First multiply each side of the equation by \( x \).

\[
x = 1 + \frac{1}{x}
\]

\[
x^2 - x - 1 = 0
\]

\[
x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)}}{2}
\]

\[
x = \frac{1 \pm \sqrt{1 + 4}}{2} \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}
\]

\[
x = \frac{1 + \sqrt{5}}{2} \text{ or } x = \frac{1 - \sqrt{5}}{2}
\]

[10](b) Simplify the expression \( (2)^{1/2} + 2(8)^{1/2} + 3(16)^{1/2} + 4(32)^{1/2} + 5(64)^{1/2} \). Do not use a calculator.

\[
\begin{align*}
\sqrt{2} &+ 2\sqrt{8} + 3\sqrt{16} + 4\sqrt{32} + 5\sqrt{64} \\
&= \sqrt{2} + 2\sqrt{2^3} + 3\sqrt{2^4} + 4\sqrt{2^5} + 5\sqrt{2^6} \\
&= \sqrt{2} + 2 \cdot 2 \cdot \sqrt{2} + 3 \cdot 2^2 \cdot \sqrt{2} + 4 \cdot 2^3 \cdot \sqrt{2} + 5 \cdot 2^4 \cdot \sqrt{2} \\
&= \sqrt{2} + 4\sqrt{2} + 12 + 16\sqrt{2} + 40 \\
&= 12 + 40 + \sqrt{2} + 16\sqrt{2} + 40 \\
&= 52 + 17\sqrt{2} + 16\sqrt{2} \\
&= 52 + 5\sqrt{2} + 16\sqrt{2} \\
&= 52 + 21\sqrt{2}
\end{align*}
\]
5. Consider the following infinite list of real numbers

\[ 0.12345678910112131415161718 \ldots \\
0.246810121416182012141618201 \ldots \\
0.369121518212427303333336 \\
0.481216202428323640444852545 \ldots \\
0.510152025303540455055606570 \ldots \\
\]

\[ \quad 6x1, 6x2, 6x3, 6x4, 6x5 \]

(5)(a) What will be the 6th number on the list

(15)(b) Using Cantor’s diagonalization argument, find a number not on the list. Justify your answer.

\[ \quad 0.612182430364248546066727884 \ldots \]

6th number on the list

b) To find a number not on the list, change one digit in each number. This will always result in a new number. To do this, change the 1st digit of the 1st number, the 2nd digit of the 2nd number... the n-th digit of the n-th number, etc. There are many ways to do this. In this case, I will change the digit to 2 when it is something other than 2; if it is 2, I will change it to 1. This results in a new number not on the list:

\[ 0.222121222 \]

I showed the diagonalization pattern on the numbers above by circling the n-th digit of the n-th number (the one that needs to be changed). Continuation of this pattern will always create a new number.