MARKS 20

[20] 1. Fibonacci Nim has 2 players and a pile of sticks. The first player takes away at least 1 but not all the sticks from the pile. The second player can take away any number from 1 to twice the number the previous player took, then the first player does the same and so on. The strategy for the starting player is to start with a non-Fibonacci number and then to write as a sum of non-consecutive Fibonacci numbers and then discard the smallest.

[8](a) Find all the Fibonacci numbers less than 400.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377

$1+1=2$
$2+3=5$
$21+34=55$
$34+55=89$
$55+89=144$
$89+144=233$
$144+233=377$

[6](b) Suppose you start with 42 sticks. What is your first move?

$42 = 34 + 8$ as a sum of nonconsecutive Fibonacci numbers. Winning strategy is to remove smallest number in this sum

* First move: remove 8 sticks

[6](c) Suppose you start with 220 sticks. What is your first move? Show your work.

$220 - 144 = 76$
$76 - 55 = 21$

$220 = 144 + 55 + 21$

First move is to remove 21 sticks, since it is the smallest number in the sum of nonconsecutive Fibonacci numbers that add to 220

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20 [20] 2. Prove that

[15] (a) $13^{1/2}$ is irrational.

[5] (b) $17 + 13^{1/2}$ is irrational, using the result from part (a).

15 a) $13^{1/2} = \sqrt{13}$

Assume that $\sqrt{13}$ is rational $\left(\frac{m}{n}\right)$, where m and n are reduced.

$\sqrt{13} = \frac{m}{n} \to (\sqrt{13})^2 = \left(\frac{m}{n}\right)^2 \to 13 = \frac{m^2}{n^2} \to 13n^2 = m^2$

** ↳ thus $m^2$ and $\underline{m}$ are both numbers with 13 as a factor

since m has 13 as a factor, it can be rewritten as $13x$

$13n^2 = (13x)^2 = (13x)(13x) = 169x^2 \to \frac{13n^2}{13} = \frac{169x^2}{13} \to n^2 = 13x^2$

** ↳ thus $n^2$ and $\underline{n}$ are both numbers with 13 as a factor

This is a contradiction to our original assumption that $\frac{m}{n}$ is reduced, since both m and n have 13 as a factor

Thus our original assumption that $\sqrt{13}$ is rational is also incorrect, so $\sqrt{13}$ must be irrational.

5 b) Assume $17 + \sqrt{13} = \frac{m}{n}$, where $\frac{m}{n}$ is a fraction in lowest terms

$\sqrt{13} = \frac{m}{n} - \frac{17}{1} \to \sqrt{13} = \frac{m}{n} - \frac{17n}{n} \to \sqrt{13} = \frac{m-17n}{n}$

↳ $\frac{m-17n}{n}$ is a fraction, since we know that $\sqrt{13}$ is irrational,

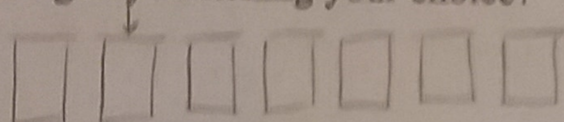
this is a contradiction.

Thus our original assumption that $17 + \sqrt{13}$ is rational is incorrect, so it must be irrational.

[20] 3. A game show requires you to randomly pick 1 of 7 envelopes. The host has hidden a $100 bill in one envelope, and a $1 bill in each of the other 6. Once you've picked, the host is required to remove 2 of the $1 envelopes you didn't pick. You now have a choice of keeping your original envelope, or paying $2 to switch your choice to one of the 4 you didn't pick.

[15](a) If you choose to pay $2 and switch your choice, then what are your odds of losing money?

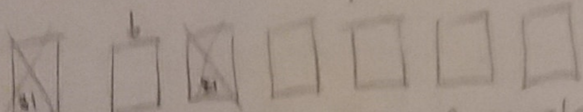
[5](b) By what percent (if any) does the odds of winning a $100. bill change by switching your choice?



a)

(losing)

- first choice: $\frac{1}{7}$ chance of winning $100, $\frac{6}{7}$ chance of winning $1



- after host removes 2 envelopes, the original selection still has a $\frac{1}{7}$ chance of winning, and the remaining envelopes still have a $\frac{6}{7}$ chance of winning, now divided between 4 envelopes instead of 6.
- each of the remaining envelopes: $\frac{1}{4}\left(\frac{6}{7}\right)$ chance of winning $= \frac{6}{28} = \frac{3}{14}$ which of course means a $\frac{11}{14}$ chance of losing $\left(1 - \frac{3}{14} = \frac{11}{14}\right)$
- If you switch envelopes, $\frac{11}{14}$ chance of opening a $1 envelope, which means an $\boxed{\frac{11}{14} \text{ chance of losing money}}$

b) As seen above, original choice has $\frac{1}{7}$ chance of winning
Also seen above: switch of choice results in $\frac{3}{14}$ chance of winning

$\left(\frac{1}{7} = \frac{2}{14}\right)$ $\frac{2}{14} \rightarrow \frac{3}{14}$ is an increase of $\frac{1}{14}$, which is 50% of $\frac{2}{14}$

↳ Thus the odds of winning a $100 bill $\boxed{\text{increase } 50\%}$ by switching choices.

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4.

[10](a) Solve the equation $x = 1 + (1/x)$ for x. Hint: First multiply each side of the equation by x.

[10](b) Simplify the expression $(2)^{1/2} + 2(8)^{1/2} + 3(16)^{1/2} + 4(32)^{1/2} + 5(64)^{1/2}$. Do not use a calculator.

$x = 1 + \frac{1}{x}$

a) $x(x) = \left(1 + \frac{1}{x}\right)x$

$x^2 = x + \frac{x}{x} \rightarrow x^2 = x + 1$

$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \rightarrow \begin{matrix} b = -1 \\ c = -1 \end{matrix}$

$x^2 - x - 1 = 0$

$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(-1)}}{2}$

$x = \frac{1 \pm \sqrt{1-(-4)}}{2} \rightarrow x = \frac{1 \pm \sqrt{1+4}}{2} \rightarrow x = \frac{1 \pm \sqrt{5}}{2}$

$\boxed{x = \frac{1+\sqrt{5}}{2} \text{ or } x = \frac{1-\sqrt{5}}{2}}$

b) $\sqrt{2} + 2\sqrt{8} + 3\sqrt{16} + 4\sqrt{32} + 5\sqrt{64}$

$= \sqrt{2} + 2\sqrt{2^2 \cdot 2} + 3\sqrt{4^2} + 4\sqrt{4^2 \cdot 2} + 5\sqrt{8^2}$

$= \sqrt{2} + (2 \cdot 2 \cdot \sqrt{2}) + (3 \cdot 4) + (4 \cdot 4 \cdot \sqrt{2}) + (5 \cdot 8)$

$= \sqrt{2} + 4\sqrt{2} + 12 + 16\sqrt{2} + 40$

$= 12 + 40 + \sqrt{2} + 4\sqrt{2} + 16\sqrt{2}$

$= 52 + \sqrt{2} + 4\sqrt{2} + 16\sqrt{2}$

$= 52 + 5\sqrt{2} + 16\sqrt{2}$

$\boxed{= 52 + 21\sqrt{2}}$

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5. Consider the following infinite list of real numbers

1) 0.123456789101112131415161718......
2) 0.246810121416182012141618201......
3) 0.369121518212427303333336
4) 0.481216202428323640444852545......
5) 0.510152025303540455055606570......
6) 6x1, 6x2, 6x3, 6x4, 6x5
..................................................

(5)(a) What will be the 6[th] number on the list

(15)(b) Using Cantor's diagonalization argument, find a number not on the list. Justify your answer.

a) 0.(6x1), (6x2), (6x3), 6x4, (6x5), ... (6xn), (6xn+1), ...

$\boxed{0.612182430364248546066727884...}$

↳ 6th number on the list

b) To find a number not on the list, change one digit in each number. This will always result in a new number. To do this, change the 1st digit of the 1st number, the 2nd digit of the 2nd number ... the nth digit of the nth number, etc. There are many ways to do this: in this case, I will change the digit to 2 when it is something other than 2; if it is 2, I will change it to 1. This results in a new number not on the list:

$\boxed{0.222121222}$

I showed the diagonalization pattern on the numbers above by circling the nth digit of the nth number (the one that needs to be changed). Continuation of this pattern will always create a new number.