Math 312: Selected Solutions to Homework 7

Section 7.1

Problem 2

Find the value of the Euler phi-function at each of these integers.

(e) $10!$

(f) $20!$

2e. We have

$10! = 1 \cdot 2 \cdot 3 \cdots 8 \cdot 9 \cdot 10.$

Factoring each term, we obtain

$10! = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5).$

Grouping the primes together, we obtain the factorization of $10!$

$10! = 2^8 \cdot 3^4 \cdot 5^2.$

Hence,

$\phi(10!) = \phi(2^8)\phi(3^4)\phi(5^2)\phi(7) = (2^7)(2 - 1)(3^3)(3 - 1)(5)(5 - 1)(7 - 1) = 829440.$

2f. We have

$20! = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5) \cdot 11 \cdot (2^2 \cdot 3) \cdot 13 \cdot (2 \cdot 7) \cdot (3 \cdot 5) \cdot 2^4 \cdot 17 \cdot (2 \cdot 3^2) \cdot 19 \cdot (2^2 \cdot 5).$

Grouping the primes together, we obtain the factorization of $20!$

$20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$

Hence,

$\phi(20!) = \phi(2^{18})\phi(3^8)\phi(5^4)\phi(7^2)\phi(11)\phi(13)\phi(17)\phi(19)
= (2^{17})(2 - 1)(3^7)(3 - 1)(5^3)(5 - 1)(7)(7 - 1)(11 - 1)(13 - 1)(17 - 1)(19 - 1)
= 416084687585280000.$
Problem 14

For which positive integers \( n \) does \( \phi(n) \mid n \)?

Suppose \( \phi(n) \mid n \) and consider the prime factorization \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \) where \( a_k \geq 1 \) and \( p_i \) are distinct primes. We have

\[
\phi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1) = p_1^{a_1-1} p_2^{a_2-1} \cdots p_k^{a_k-1} = n.
\]

Clearly if \( n = 1 \) we have \( \phi(1) = 1 \) hence \( \phi(n) \mid n \) holds trivially. Similarly, if \( n = 2 \) we have \( \phi(n) = 1 \) and therefore \( \phi(n) \mid n \).

Suppose now that \( n > 2 \). Since \( n \) now must contain a prime factor larger than 2, by the formula

\[
\phi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1),
\]

we observe that \( \phi(n) \) is necessarily even. Now, since \( \phi(n) \mid n \), this means that \( p_1 = 2 \) must appear as a factor of \( n \). That is, \( n = 2^{a_1} p_2^{a_2} \cdots p_k^{a_k} \).

Suppose further that \( n > 2 \) has two odd prime factors, \( p_2, p_3 \). If \( a_2 > 0 \) and \( a_3 > 0 \), then \( p_2 - 1 \) and \( p_3 - 1 \) are even and so \( \phi(n) \) has two factors of 2 as well as the factor \( 2^{a_1-1} \). That is, \( \phi(n) \) has a factor of \( 2^{a_1-1+2} = 2^{a_1+1} \). Since \( \phi(n) \mid n \), this means \( 2^{a_1+1} \mid n \), a contradiction since \( n = 2^{a_1} p_2^{a_2} \cdots p_k^{a_k} \).

It follows that \( n = 2^{a_1} p_2^{a_2} \) for some prime \( p_2 > 2 \). Now, \( \phi(n) = 2^{a_1-1} p_2^{a_2-1}(p_2 - 1) \), where \( p_2 - 1 \) is even. Since \( \phi(n) \mid n \), the only prime factors which can show up in \( \phi(n) \) are 2 and \( p_2 \). This means that \( p_2 - 1 \) must be a power of 2, say \( p_2 - 1 = 2^l \) for some \( l \). Now,

\[
\phi(n) = 2^{a_1-1} p_2^{a_2-1}(p_2 - 1) = 2^{a_1-1+l} p_2^{a_2-1} | 2^{a_1} p_2^{a_2}.
\]

This forces \( l = 1 \) and \( p_2 = 3 \). That is, \( n = 2^{a_1} 3^{a_2} \), where \( a_2 \geq 0 \) and \( a_3 \geq 0 \).

Section 7.2

Problem 2

Find the number of positive integer divisors of each of these integers.

(e) \( 2 \cdot 3^2 \cdot 5^3 \cdot 7^4 \cdot 11^5 \cdot 13^4 \cdot 17^5 \cdot 19^5 \)

(f) 20!
2e. The number of positive integer divisors is denoted by the function $\tau$, which is a multiplicative function such that $\tau(p^a) = a + 1$. It follows that

$$\tau(2 \cdot 3^2 \cdot 5^3 \cdot 7^4 \cdot 11^5 \cdot 13^4 \cdot 17^5 \cdot 19^5) = \tau(2)\tau(3^2)\tau(5^3)\tau(7^4)\tau(11^5)\tau(13^4)\tau(17^5)\tau(19^5)$$

$$= (1 + 1) \cdot (2 + 1) \cdot (3 + 1) \cdot (4 + 1) \cdot (5 + 1) \cdot (4 + 1) \cdot (5 + 1) \cdot (5 + 1)$$

$$= 129600.$$

2f. The number of positive integer divisors is denoted by the function $\tau$, which is a multiplicative function such that $\tau(p^a) = a + 1$. Now

$$20! = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 \cdot 3^2 \cdot (2 \cdot 5) \cdot 11 \cdot (2^2 \cdot 3) \cdot 13 \cdot (2 \cdot 7) \cdot (3 \cdot 5) \cdot 2^4 \cdot 17 \cdot (2 \cdot 3^2) \cdot 19 \cdot (2^2 \cdot 5).$$

Grouping the primes together, we obtain the factorization of $20!$

$$20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$$

It follows that

$$\tau(20!) = \tau(2^{18})\tau(3^8)\tau(5^4)\tau(7^2)\tau(11)\tau(13)\tau(17)\tau(19)$$

$$= (18 + 1) \cdot (8 + 1) \cdot (4 + 1) \cdot (2 + 1) \cdot (1 + 1) \cdot (1 + 1) \cdot (1 + 1) \cdot (1 + 1)$$

$$= 41040.$$

Problem 4

For which positive integers $n$ is the sum of divisors of $n$ odd?

Let $n \in \mathbb{Z}_{>0}$ and consider its prime decomposition $n = 2^d p_1^{d_1} \cdots p_r^{d_r}$, where $p_i$ are distinct odd primes. As $\sigma$ is multiplicative, we have

$$\sigma(n) = \sigma(2^d)\sigma(p_1^{d_1}) \cdots \sigma(p_r^{d_r}).$$

Thus $\sigma(n)$ is odd if and only if all its factors above, which are of the form $\sigma(p^k)$ where $p$ is a prime, are odd. For any prime $p$ we have $\sigma(p^k) = 1 + p + \cdots + p^k$ which is odd if and only if $p + \cdots + p^k$ is even. This is the case when $p = 2$ or if $p$ is odd but we have an even number of odd terms in the sum, that is $k$ even.

Thus $\sigma(n)$ is odd if and only if each odd prime $p$ dividing $n$ occurs with an even exponent in the prime factorization of $n$. That is, the sum of the divisors of $n$ is odd if and only if $n$ is of the form $n = 2^d p_1^{d_1} \cdots p_r^{d_r}$ with $d_i = 2d_i'$ for all $i$. Equivalently, when $n$ is of the form $2^d m^2$ for some odd integer $m$.

Problem 6

Find the smallest positive integer $n$ with $\tau(n)$ equal to each of these integers.

(e) 14

(f) 100
6e. If \( n = p_1^{a_1} \cdots p_k^{a_k} \) is the prime factorization of \( n \), then

\[
14 = \tau(n) = (a_1 + 1) \cdot (a_2 + 1) \cdots (a_k + 1).
\]

Note that 14 factors into a product of positive integers in only two ways,

\[
14 = 1 \cdot 14 = 2 \cdot 7.
\]

This corresponds to \( k = 1 \) and \( a_1 = 13 \) or to \( k = 2 \) and \( a_1 = 6 \) and \( a_2 = 1 \). Hence

\[
n = p_1^{13} \quad \text{or} \quad n = p_1^6 p_2
\]

where \( p_1, p_2 \) are different primes. The smallest solution in each case is

\[
n = 2^{13} = 8192 \quad \text{or} \quad n = 2^6 \cdot 3 = 192.
\]

Hence the smallest integer such that \( \tau(n) = 14 \) is \( n = 192 \).

6f. If \( n = p_1^{a_1} \cdots p_k^{a_k} \) is the prime factorization of \( n \), then

\[
100 = \tau(n) = (a_1 + 1) \cdot (a_2 + 1) \cdots (a_k + 1).
\]

Note that 100 factors into a product of positive integers in only 9 ways,

(1) \( 2 \cdot 2 \cdot 5 \cdot 5 \)
(2) \( 2 \cdot 2 \cdot 25 \)
(3) \( 4 \cdot 5 \cdot 5 \)
(4) \( 2 \cdot 5 \cdot 10 \)
(5) \( 2 \cdot 50 \)
(6) \( 4 \cdot 25 \)
(7) \( 5 \cdot 20 \)
(8) \( 10 \cdot 10 \)
(9) \( 1 \cdot 100 \)

Respectively, this corresponds to

(1) \( k = 4 \) and \( a_1 = 1, a_2 = 1, a_3 = 4, a_4 = 4 \). Hence \( n = p_1 p_2 p_3^4 p_4^2 \)
(2) \( k = 3 \) and \( a_1 = 1, a_2 = 1, a_3 = 24 \). Hence \( n = p_1 p_2 p_3^{24} \)
(3) \( k = 3 \) and \( a_1 = 3, a_2 = 4, a_3 = 4 \). Hence \( n = p_1^3 p_2^2 p_3^4 \)
(4) \( k = 3 \) and \( a_1 = 1, a_2 = 4, a_3 = 9 \). Hence \( n = p_1 p_2^2 p_3^6 \)
(5) \( k = 2 \) and \( a_1 = 1, a_2 = 49 \). Hence \( n = p_1 p_2^{49} \)
(6) \( k = 2 \) and \( a_1 = 3, a_2 = 24 \). Hence \( n = p_1^3 p_2^{24} \)
(7) \( k = 2 \) and \( a_1 = 4, a_2 = 19 \). Hence \( n = p_1^4 p_2^{19} \)
(8) \( k = 2 \) and \( a_1 = 9, a_2 = 9 \). Hence \( n = p_1^3 p_2^9 \)
(9) \( k = 1 \) and \( a_1 = 99 \). Hence \( n = p_1^{99} \)

Here, \( p_1, p_2, p_3, p_4 \) are different primes. The smallest solution in each case is

(1) \( n = 2^4 \cdot 3^4 \cdot 5 \cdot 7 = 45360. \)
(2) \( n = 2^6 \cdot 3 \cdot 5 = 25165824. \)
(3) \( n = 2^4 \cdot 3^4 \cdot 5^3 = 162000. \)
(4) \( n = 2^9 \cdot 3^4 \cdot 5 = 207360. \)
(5) \( n = 2^{49} \cdot 3. \)
(6) \( n = 2^{24} \cdot 3^4 = 452984832. \)
(7) \( n = 2^{19} \cdot 3^4 = 42467328. \)
(8) \( n = 2^9 \cdot 3^4 \cdot 5 = 10077696. \)
(9) \( n = 2^{99}. \)

Hence the smallest integer such that \( \tau(n) = 100 \) is \( n = 45360 \).
Section 7.3

Problem 18

Show that $30240 = 2^5 \cdot 3^3 \cdot 5 \cdot 7$ is 4-perfect.

Recall that an integer $n$ is called $k$-perfect if $\sigma(n) = kn$. Recall further that $\sigma(p^a) = \frac{p^{a+1}-1}{p-1}$ where $p$ is prime. Hence, we compute

\[
\sigma(30240) = \sigma(2^5)\sigma(3^3)\sigma(5)\sigma(7)
= \frac{2^6 - 1}{2 - 1} \cdot \frac{3^4 - 1}{3 - 1} \cdot \frac{5^2 - 1}{5 - 1} \cdot \frac{7^2 - 1}{7 - 1}
= 63 \cdot 40 \cdot 6 \cdot 8
= 120960
= 4 \cdot 30240.
\]