Math 312: Selected Solutions to Homework 2

Section 1.4

Problem 6

Prove that $f_{n-2} + f_{n+2} = 3f_n$ whenever $n$ is an integer with $n \geq 2$. (Recall that $f_0 = 0$.)

Let $n \geq 2$. Recall that for $n \geq 3$, we have

$$f_n = f_{n-1} + f_{n-2}$$

by definition of the Fibonacci sequence. If we let $f_0 = 0$, we can extend the above recurrence relation to $n = 2$. That is, for $n \geq 2$, we have

$$f_n = f_{n-1} + f_{n-2}.$$ 

In particular, if $n \geq 2$, then of course $n+2 \geq 4 > 2$ so that $f_{n+2} = f_{n+1} + f_n$ holds. Similarly, $n + 1 \geq 3 > 2$ so that $f_{n+1} = f_n + f_{n-1}$ holds. Now,

$$f_{n-2} + f_{n+2} = f_{n-2} + (f_{n+1} + f_n) \quad \text{by our first observation on } f_{n+2}$$

$$= f_{n-2} + ((f_n + f_{n-1}) + f_n) \quad \text{by our second observation on } f_{n+1}$$

$$= 2f_n + f_{n-1} + f_{n-2}$$

$$= 2f_n + f_n \quad \text{by definition of the Fibonacci sequence}$$

$$= 3f_n.$$ 

Section 1.5

Problem 36

Show that if $a$ is an integer, then 3 divides $a^3 - a$.

Let $a \in \mathbb{Z}$. Dividing $a$ by 3 we get $a = 3q + r$ with $r = 0, 1, 2$. Note that

$$a^3 - a = (a - 1)a(a + 1) = (3q + r - 1)(3q + r)(3q + r + 1)$$

and clearly for any choice of $r = 0, 1, 2$ one of the three factors is a multiple of 3. This is the same as saying that in among three consecutive integers one must be a multiple of 3.
Section 2.1

Problem 4

Convert \((101001000)_2\) from binary to decimal notation and \((1984)_{10}\) from decimal to binary notation.

To convert \((101001000)_2\) from binary to decimal notation, we have
\[
101001000 = 0 \cdot 2^0 + 0 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 + 0 \cdot 2^5 + 1 \cdot 2^6 + 1 \cdot 2^8 = (328)_{10}.
\]

To convert \((1984)_{10}\) from decimal to binary notation, we have
\[
\begin{align*}
1984 &= 2 \cdot 992 + 0, \\
992 &= 2 \cdot 496 + 0, \\
496 &= 2 \cdot 248 + 0, \\
248 &= 2 \cdot 124 + 0, \\
124 &= 2 \cdot 62 + 0, \\
62 &= 2 \cdot 31 + 0, \\
31 &= 2 \cdot 15 + 1, \\
15 &= 2 \cdot 7 + 1, \\
7 &= 2 \cdot 3 + 1, \\
3 &= 2 \cdot 1 + 1, \\
1 &= 2 \cdot 0 + 1.
\end{align*}
\]

To obtain the base 2 expansion of 1984, we simply take the remainders of these divisions. This shows that
\((1984)_{10} = (11111000000)_2\).

Section 2.2

Problem 10

Subtract \((CAFE)_{16}\) from \((FEED)_{16}\)

To subtract \((CAFE)_{16}\) from \((FEED)_{16}\), we have
\[
\begin{array}{cccc}
\text{Borrow Row:} & -1 & -1 \\
\text{First Number:} & F & E & E & D \\
\text{Second Number:} & C & A & F & E \\
\hline
\text{Difference:} & 3 & 3 & E & F.
\end{array}
\]

To obtain the above, we performed the following operations:
1. **D-E.** We compute \(D - E = 13 - 14 = -1 = -1 \cdot 16 + 15 = -1 \cdot 15 + F\), hence \(B_0 = -1\) and \(d_0 = F\).
2. **E-F.** We compute \(E - F + B_0 = 14 - 15 - 1 = -2 = -1 \cdot 16 + 14 = -1 \cdot 16 + E\), hence \(B_1 = -1\) and \(d_1 = E\).
3. **E-A.** We compute \(E - A + B_1 = 14 - 10 - 1 = 3 = 0 \cdot 16 + 3\), hence \(B_1 = 0\) and \(d_2 = 3\).
4. **F-C.** We compute \(F - C + B_2 = 15 - 12 + 0 = 3 = 0 \cdot 16 + 3\), hence \(B_2 = 0\) and \(d_3 = 3\).
Section 3.1

Problem 8

(This exercise constructs another proof of the infinitude of primes.) Show that the integer \( Q_n = n! + 1 \), where \( n \) is a positive integer, has a prime divisor greater than \( n \). Conclude that there are infinitely many primes.

Let \( n \in \mathbb{Z}_{>0} \). Consider \( Q_n = n! + 1 \). There is a prime factor \( p \mid Q_n \). Suppose \( p \leq n \); then \( p \mid n! = n(n-1)(n-2) \cdots 2 \cdot 1 \) therefore \( p \mid Q_n - n! = 1 \), a contradiction. We conclude that \( p > n \). In particular, given a positive integer \( n \) we can always find a prime larger than \( n \); by growing \( n \) we produce infinitely many arbitrarily large primes.