1.2 Sums and Products

\[ \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n \]

\[ \prod_{j=1}^{n} a_j = a_1 a_2 a_3 \ldots a_n \]

Example \[ \sum_{j=0}^{5} j! = 0! + 1! + 2! + 3! + 4! + 5! \]

\[ = 1 + 1 + 2(1) + 3(2)(1) + 4(3)(2)(1) + 5(4)(3)(2)(1) \]

\[ = 4 + 6 + 24 + 120 = 154 \]

A function \( f : \mathbb{N} \rightarrow \mathbb{N} \) is defined recursively if \( f(1) \) is given and for each positive integer \( n \) a rule is given for determining \( f(n+1) \) from \( f(n) \).

Definition \( f(n) = n! \) is defined recursively by \( f(1) = 1! = 1 \) and \( f(n+1) = (n+1)! \) \( f(n) \).

Extend domain to include 0 and define \( f(0) = 0! = 1 \).
#19 022 List the integers 100!, 100^{100}, 2^{100} and (50!)^2 in order of size. Justify answer.

\[ 100^{100} = 100 \cdot 100 \cdot 100 \cdots (100) \]
\[ 100! = 100 \cdot 99 \cdot 98 \cdots (51)50 \cdots (1) \]
\[ (50!)^2 = 50 \cdot 49 \cdot 48 \cdots 1 \cdot 50 \cdot 49 \cdots (17) \]
\[ 2^{100} = 2 \cdot 2 \cdot 2 \cdots (2) \]

CLAIM 1.1 \[ \sum_{j=0}^{n} a r^j = a \left( \frac{r^{n+1} - 1}{r-1} \right) \]

Proof: \[ (a + ar + ar^2 + \cdots ar^n)(r-1) \]
\[ = ar + ar^2 + ar^3 + \cdots ar^{n+1} - (a + ar + ar^2 + \cdots ar^n) \]
\[ = ar^{n+1} - a \quad \text{Thus} \quad \sum_{j=0}^{n} a r^j (r-1) = a(r^{n+1} - 1) \]

and the result follows.

CLAIM 2 \[ \sum_{j=0}^{\infty} a r^j = \lim_{n \to \infty} \sum_{j=0}^{n} a r^j \]
\[ = \lim_{n \to \infty} a \left( \frac{r^{n+1} - 1}{r-1} \right) \]
\[ = \frac{a}{1-r} \quad \text{if} \quad |r| < 1 \]

Example: \[ T = \frac{1}{1} + \frac{1}{10} + \left( \frac{1}{10} \right)^2 + \cdots + \left( \frac{1}{10} \right)^n + \cdots \]
\[ \frac{1}{1-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = 1 \frac{1}{9} = 1 + \frac{1}{9} \]
Triangular numbers \( t_n \) count the dots in the following triangles.

\[
\begin{align*}
  t_1 &= 1 \\
  t_2 &= 3 \\
  t_3 &= 6 \\
  t_4 &= 10
\end{align*}
\]

Thus \( t_n = 1 + 2 + 3 + \ldots + n \)

**Claim** \( t_n = \frac{n(n+1)}{2} \)

If \( n \) is even then there are \( \frac{n}{2} \) pairs adding to \( n+1 \)

\[
\frac{n}{2}(n+1) = \frac{n(n+1)}{2}
\]

If \( n \) is odd and \( >1 \)

Then \( t_n = t_{n-1} + n = \frac{(n-1)(n-1+1)}{2} + n \)

\[
= \frac{(n-1)n}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2}
\]

\[
= \frac{n(n+1)}{2}, \quad n=5 \quad 1 + 2 + 3 + 4 + 5
\]

\[
= 5(2) + 5 = \frac{4(5)}{2} + 5
\]

\[
= \frac{4(5) + 2(5)}{2} = \frac{5(6)}{2}
\]
1.3 Mathematical Induction

Problem 1 page 27

Use mathematical induction to show that

\[ A_n \triangleq n < 2^n \text{ for all positive integers} \]

Proof: Let \( S \) be the set of all positive integers for which \( A_n \) is true.

\[ A_1 : 1 < 2 \quad \text{is true. Thus } 1 \in S \]

Suppose \( A_n : n < 2^n \) is true some \( n \).

Then \( A_{n+1} : n+1 < 2^{n+1} \) must be true because \( n+1 < 2^n + 1 \)

\[ < 2^n + 2^n = 2^{n+1}. \] Thus whenever \( n \in S \) then \( n+1 \in S \). The states

Principle of Induction states that \( S \) contains all integers, that is, all integers \( n \)

\( A_n \) is true.\]
Induction (1st Principal). If $S$ is a set of positive integers containing 1 and also has the property that whenever it contains the integer $n$ it contains the integer $n+1$, then $S = \mathbb{N}$ the set of all positive integers.

Induction (2nd Principal = strong induction) it is the case that if $S$ contains 1 and if whenever it contains $n$ it also contains 1, 2, ..., $n$ then it contains $n+1$, then it must contain all positive integers.

Simple example from page 25

Any amount of postage > 1 cent can be made from 2 and 3 cent stamps

$n = 2 \quad 1 \text{ 2-cent stamp}$

$n = 3 \quad 1 \text{ 3-cent stamp}$

assumption for fixed $n \geq 3$ all amounts $2, 3, ..., n$ can be formed as required
to show you can do the same for \( n+1 \) solution - use a 2 cent stamp plus the combination required to make \( n-1 \) cents.

Point is you don't have to use the truth for \( n \) to prove \( n+1 \), you are allowed to use values smaller than \( n \) as well!

**EXAMPLE** Prove that \[ \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \]

for all \( n \).

**Solution** \[ \sum_{j=1}^{n} j^2 = 1^2 = 1 = \frac{1(1+1)(2+1)}{6} \]

\[ = \frac{1(2)(3)}{6} = 1 \]

Suppose true for fixed \( n \) that \[ \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \] then we must prove \[ \sum_{j=1}^{n+1} j^2 = \frac{(n+1)(n+2)(2n+3)}{6} \]

**Proof** \[ \sum_{j=1}^{n+1} j^2 = \sum_{j=1}^{n} j^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \]

\[ = (n+1) \left( \frac{n(2n+1)}{6} + \frac{6(n+1)}{6} \right) = \frac{n+1}{6} \left( 2n^2 + 7n + 6 \right) \]

\[ = \frac{n+1}{6} \left( 2n^2 + 7n + 6 \right) = \frac{n+1}{6} \left( 2(n+3)(n+2) \right) = \frac{(n+1)(n+2)(2n+3)}{6} \]

as required