CHAPTER 1  The Integers

1.1 Numbers and Sequences

Natural numbers

\[ N = \{1, 2, 3, 4, \ldots, n, n+1, \ldots\} \]

Integers

\[ \mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, 2, 3, \ldots\} \]

Rationals

\[ \mathbb{Q} = \{ r \mid r = \frac{p}{q} \text{ for integers } p \text{ and } q, q \neq 0\} \]

Real numbers

\[ \mathbb{R} = \text{ The set of all real numbers} \]

A sequence in a set \( A \) is a function

\[ f : \mathbb{N} \rightarrow A \]  or \[ a_1, a_2, \ldots, a_n, a_{n+1}, \ldots \]

where \( a_n = f(n) \).
Well-Ordering Property

Every nonempty set of positive integers has a least element.

This property can be used to show that $\sqrt{2}$ is irrational, that is, not rational.

See Theorem 1.1 page 6.

Here I will give a different proof that $\sqrt{2}$ is irrational.

Suppose $\sqrt{2} = \frac{a}{b}$ where $a$ and $b$ are integers with no common factors and $b \neq 0$.

Then $2 = \frac{a^2}{b^2}$ so that $2b^2 = a^2$. Then 2 divides $a^2$, thus 2 divides $a$ since a odd would imply $a^2$ odd. Thus $a = 2c$ for an integer $c$. Thus $2b^2 = 2c \cdot a^2 = (2c)^2 = 4c^2$ and $b^2 = 2c^2$, thus 2 divides $b^2$, hence 2 divides $b$. Thus $a$ and $b$ have common factor 2. Contradiction.
The greatest integer in a real number $x$, denoted by $\lfloor x \rfloor$, is the largest integer $\leq x$.

Examples: $\lfloor \frac{7}{2} \rfloor = 3$ and $\lfloor -\frac{7}{2} \rfloor = -4$

The fractional part of $x$, denoted by $\{x\}$

$= x - \lfloor x \rfloor < 1$

**Diophantine approximation** — approximates real numbers by rationals.

**Pigeonhole principle** If $k+1$ objects are placed into $k$ boxes, then at least one box contains 2 or more objects.

**Theorem 1.3**  
**Dirichlet approximation** If $x$ is a real number and $n$ a positive integer, then there exist integers $a$ and $b$ with $1 \leq a \leq n$ such that $|ax - b| < \frac{1}{n}$

Read the general proof on page 9

I will give an interpretation and prove a special case.
Simplified interpretation of Theorem 1.3

At least one of the numbers \( \alpha, 2\alpha, 3\alpha, 4\alpha, \ldots \) is within distance \( \frac{1}{n} \) of an integer for any real number \( \alpha \) and any positive integer \( n \).

Proof of special case when \( 0 \leq \alpha \leq 1 \) and \( n = 3 \). We must show that at least one of \( \alpha, 2\alpha, 3\alpha \) is within distance \( \frac{1}{3} \) of an integer for all real \( \alpha \) in the closed interval \([0,1] \).

\[
\begin{array}{cccc}
0 & \frac{1}{3} & \frac{2}{3} & 1 \\
\end{array}
\]

Case 1. If \( \alpha \in [0, \frac{1}{3}) \) then \( \alpha < \frac{1}{3} \) from 0.

If \( \alpha \in (\frac{2}{3}, 1] \) then \( \alpha < \frac{1}{3} \) from 1.

Case 2. If \( \alpha \in (\frac{1}{3}, \frac{2}{3}) \) then \( 2\alpha \in (\frac{2}{3}, \frac{4}{3}) \)

and \( 2\alpha < \frac{1}{3} \) unit from 1.

Case 3. If \( \alpha = \frac{1}{3} \) then \( 3\alpha = 1 \) is \( 0 < \frac{1}{3} \) unit from 1.

If \( \alpha = \frac{2}{3} \) then \( 3\alpha = 2 \) is \( 0 < \frac{1}{3} \) unit from 2.
A set is countable if it is finite or it is infinite and there is a 1-1 correspondence between the set and the set of positive integers.

Examples
1. The set of rationals is countable — see page 11
2. The set of real numbers is not countable — see problem 4.5 page 15.

Sample of Exercises

Easy
1. p12 — which of following are well-ordered?
   a. Integers > 3 — Yes
   b. Positive even integers — Y
   c. Positive rationals — N
   d. Positive rationals of form \( \frac{a}{2} \) for \( a \), a positive integer
   e. Nonnegative rationals — N

Difficult
19. Show \( \lfloor \sqrt[3]{x} \rfloor = \lfloor \sqrt{x} \rfloor \) whenever \( x \) is nonnegative real.

Proof. Let \( x = k + \epsilon \) where \( k \) is an integer and \( 0 \leq \epsilon < 1 \). Let \( k = a^2 + b \) where \( a \) is the largest integer with \( a^2 \leq k \). Then \( a^2 \leq k = a^2 + b = x = a^2 + b + \epsilon \leq (a+1)^2 \).
Then \( \lfloor \sqrt{x} \rfloor = a \) and \( \lfloor \sqrt[3]{x} \rfloor = \lfloor \sqrt{k} \rfloor = a \) also