the Parthenon was built, it was much fancier—in particular, it had a roof. Imagine now that the roof is in place. If we form the rectangle from the tip of the rooftop to the steps, we will see a nearly perfect Golden Rectangle.

Another example of the Golden Rectangle in Greek sculpture is the Grecian eye cup. The one pictured is inscribed inside a perfect Golden Rectangle.

It remains an unanswered question whether Greek artists and designers intentionally used the Golden Rectangle in their work or chose those dimensions solely based on aesthetic tastes. In fact, we are not even certain that such artists were consciously aware of the Golden Rectangle. Although we will likely never know the truth, it is romantic to hypothesize that the Greeks were not conscious of the Golden Rectangle, because this then shows how aesthetically appealing its dimensions are and that we are naturally attracted to such shapes. Some people, however, believe that the occurrence of Golden Rectangle proportions is simply coincidental and random. While some believe that ancient Greek works definitely contain Golden Rectangles, others believe that it is nearly impossible to measure such works or ruins accurately; thus, there is plenty of room for error. In the preceding pictures, all the superimposed rectangles are perfect Golden Rectangles. Was their presence random or deliberate? Are Golden Rectangles really there? What do you think?
The Golden Rectangle in the Renaissance

It appears that mathematicians in the Middle Ages and the Renaissance were fascinated by the Golden Rectangle, but there is much question as to whether this enthusiasm was shared by artists of the time. Leonardo da Vinci was a math enthusiast, but did he know about the Golden Rectangle? Did he deliberately use it in his work? While historians debate such issues, let's take a look at Leonardo's unfinished portrait of St. Jerome from 1483. In the reproduction on below, we have superimposed a perfect Golden Rectangle around the great scholar's body.

Intentionally or otherwise, Leonardo selected proportions that were aesthetically appealing, and such dimensions resemble those of the Golden Rectangle. Although we are not certain whether Leonardo intentionally used the Golden Rectangle, we do know that 26 years later he was aware of its existence. In 1509, Leonardo was the illustrator for Luca Pacioli's text on the Golden Ratio titled *De Divina Proporzione*. It was famous mainly for the reproductions of 60 geometrical drawings illustrating the Golden Ratio.

The *Divine Proportion* is a synonym for the Golden Ratio. In fact, many people, including Johannes Kepler, referred to the Golden Ratio as the Divine Proportion, or as the *Mean and Extreme Ratio*. Sometimes imaginations ran a bit too wild. Pacioli claimed that one's belly button divides one's body into the Divine Proportion. If you're not ticklish, you can easily check that this is not necessarily true.
The Golden Rectangle and Impressionism

Let's now leap ahead about 300 years to the creative age of French Impressionism. Painter Georges Seurat was captivated by the aesthetic appeal of the Golden Ratio and the Golden Rectangle. In his painting *La Parade* from 1888, he carefully planted numerous occurrences of the Golden Ratio through the positions of the people and the delineation of the colors. The use of the Golden Ratio in works of art is now known as the technique of dynamic symmetry.
The Golden Rectangle in the 20th Century

In the 20th century, artists were still fascinated with the beautiful proportions of the Golden Rectangle. French architect Le Corbusier believed that people are comforted by mathematics. In this spirit, he deliberately designed this villa to conform with the Golden Rectangle.

Le Corbusier was one of the architects involved in the design of the United Nations Headquarters in New York City. Here we again see the influence of the Golden Rectangle in this monolithic structure.
1.5 Divisibility - Main Points

Suppose $a$ and $b$ are integers with $b \neq 0$
then write $a \mid b$ if there is an integer $c$
with $b = ac$. Write $a \notmid b$ if such an
integer does not exist.

Example: $13 \mid 52$ since $52 = 13(4)$

but $13 \nmid 54$ since $54 = 13(4) + 2$

The notation $a \mid b$ should not be
confused with $a/b$ which is the
rational number obtained when
$a$ is divided by $b$ ($a, b$ integers $b \neq 0$)

Division algorithm $a, b$ integers, $b > 0$
then there are unique integers
$q$ and $r$ with $a = bq + r$ $0 \leq r < b$

Example $a = 19$, $b = 3$ then $q = 6$, $r = 1$

\[ 3 \left[ \frac{19}{18} \right] \] and \[ 19 = 3(6) + 1 \]
FORMULAS for $a$ and $r$


g = \left\lfloor \frac{a}{b} \right\rfloor \text{ and } r = a - b \left\lfloor \frac{a}{b} \right\rfloor

greatest integer

Example: \( a = 19 \), \( b = 3 \), \( \frac{a}{b} = \frac{19}{3} \)

\[ \left\lfloor \frac{19}{3} \right\rfloor = 6 \text{ and } r = 19 - 3 \left\lfloor \frac{19}{3} \right\rfloor = 19 - 3(6) = 1 \]

\[ 19 = 3(6) + 1 \]

Example: \( a = 1717 \), \( b = 7 \)

\[ \frac{1717}{7} = 245 \frac{2}{7} \]

Then \( \left\lfloor \frac{1717}{7} \right\rfloor = 245 = 8 \)

\[ \frac{245}{7} = 35 \frac{0}{7} \]

\[ r = 1717 - 7 \left\lfloor \frac{1717}{7} \right\rfloor = 1717 - 7(245) = 1717 - 1715 = 2 \]

Then \( a = 6g + r \) is \( 1717 = 7(245) + 2 \)

#38 page 41. Show that the square of every odd integer is of the form \( 8k + 1 \)

Proof: An odd integer is of the form \( 2n + 1 \), \( (2n+1)^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1 \)

But \( n^2 + n \) is even since \( n \) even imples \( n^2 \) and \( n^2 + n \) even, and \( n \) odd implies \( n^2 \) odd and hence \( n^2 + n \) even. Thus \( n^2 + n = 2k \)
Chapter 2  Integer Representations and operations

OMIT 2.3

2.1 Representations of integers to different bases.

Let \( b \) be a positive integer \( > 1 \)

**Theorem** Every positive integer \( n \) can be written uniquely in the form

\[
N = a_k b^k + a_{k-1} b^{k-1} + \ldots + a_1 b + a_0
\]

for nonnegative \( k \) where \( 0 \leq a_j \leq b-1 \)

\( a_k \neq 0 \) and \( 0 \leq a_j \leq b-1 \) for \( j = 0, 1, \ldots, k \)

**Notation:** We write \( N = (a_k a_{k-1} \ldots a_1 a_0)_b \)

base \( b \) requires \( b \) single digits

**Examples**

- \( b = 10 \) requires 10 digits 0, 1, 2, \ldots, 9 as usual
- \( b = 7 \) requires 7 digits 0, 1, 2, \ldots, 6
- **Binary** \( b = 2 \) requires 2 digits 0, 1
- **Hexadecimal** \( b = 16 \) requires 16 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F
EXAMPLE \[ 27 = 2(10) + 7 = (27)_{10} \]
\[ 27 = 3(7) + 6(1) = (36)_7 \]
\[ 27 = 2^4 + 2^3 + 2^1 + 2^0 = (11011)_2 \]
\[ 27 = 16^1 + 516^0 = (1B)_{16} \]

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Convert \((1999)_{10}\) from decimal to base 7

\((1999)_{10} = 1 \times 10^3 + 9 \times 10^2 + 9 \times 10 + 9\)

**Solution**

\[ 7^3 = 343 \]
\[ \left[ \frac{1999}{343} \right] = 5 = \text{largest multiple of } 7^3 \text{ in } 1999 \]

\[ 5 \times 7^3 = 1715 \]
\[ 1999 - 1715 = 284 \]

\[ 7^2 = 49 \]
\[ \left[ \frac{284}{49} \right] = 5 \]

\[ 5 \times 7^2 = 245 \]
\[ 284 - 245 = 39 \]

\[ \left[ \frac{39}{7} \right] = 5 \]

\[ 5 \times 7 = 35 \]
\[ 39 - 35 = 4 \]

**Thus** \((1999)_{10} = 1715 + 245 + 35 + 4\)

\[ = 5 \times 7^3 + 5 \times 7^2 + 5 \times 7 + 4 \]

\[ = (5554)_7 \]