LAPLACE TRANSFORMS

WE DEFINE THE LAPLACE TRANSFORM BY

$$\hat{F}(s) = \int_0^\infty e^{-st} f(t)\,dt = \mathcal{L}(f(t)).$$

WE REQUIRE $|f(t)| \le K e^{\alpha t}$ FOR ALL t, AND $f(t)$ IS PIECEWISE CONTINUOUS.

THEN, WE OBTAIN THAT $F(s)$ IS ANALYTIC FOR $\mathrm{RE}(s) > \alpha$.

PROPERTIES

(i) $\mathcal{L}(f_1 + f_2) = \mathcal{L}(f_1) + \mathcal{L}(f_2)$ LINEARITY

(ii) $\mathcal{L}(f'') = s^2 \mathcal{L}(f) - f'(0) - s f(0)$. $\quad \mathcal{L}(f') = s\mathcal{L}(f) - f(0)$.

(iii) $\mathcal{L}(f(t)e^{at}) = F(s-a)$, WHERE $F(s) = \mathcal{L}(f(t))$.

(iv) $\mathcal{L}(t f(t)) = \int_0^\infty t f(t) e^{-st}\,dt = -F'(s)$, WHERE $F(s) = \mathcal{L}(f(t))$.

(v) $f(t+T) = f(t)$. THEN,

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st}\,dt = \int_0^T f(t) e^{-st}\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \dots + \int_{(N-1)t}^{Nt} f(t) e^{-st}\,dt + \dots$$

$$= \int_0^T f e^{-st}\,dt \left(1 + e^{-sT} + e^{-2sT} + \dots\right)$$

HENCE,

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

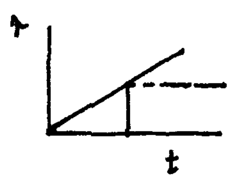
(vi) LET $u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$ UNIT STEP FUNCTION.

THEN, $\mathcal{L}(u_c(t) f(t-c)) = \int_c^\infty e^{-st} f(t-c)\,dt = e^{-sc} F(s)$.

(vii) CONVOLUTION $\quad (g * h)(t) = \int_0^t g(t-\tau) h(\tau)\,d\tau = f(t)$.

THEN, $\mathcal{L}[(g*h)(t)] = \hat{G}(s)\hat{H}(s)$. $\quad \hat{G}(s) = \mathcal{L}(g)$, $\hat{H}(s) = \mathcal{L}(h)$.

PROOF $\quad \hat{F}(s) = \int_0^\infty e^{-st}\left(\int_0^t g(t-\tau) h(\tau)\,d\tau\right)dt$



NOW INTERCHANGING ORDER OF INTEGRATION,

$$\hat{F}(s) = \int_0^\infty h(\tau) \int_\tau^\infty e^{-st} g(t-\tau)\,dt\,d\tau = \int_0^\infty h(\tau) e^{-s\tau} \left(\int_\tau^\infty e^{-s(t-\tau)} g(t-\tau)\,dt\right) d\tau$$

LETTING $u = t - \tau$, $du = dt$, WE GET $\hat{F}(s) = \hat{G}(s)\hat{H}(s)$

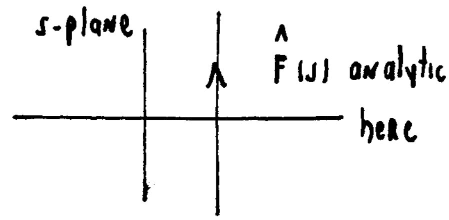
(viii) **INVERSION FORMULA**

LET $\hat{F}(s) = \int_0^\infty e^{-st} f(t)\,dt.$

SUPPOSE THAT THE INTEGRAL CONVERGES FOR SOME s_0, THEN IT CONVERGES FOR ALL s WITH $RE(s) > RE(s_0)$; IN THIS RANGE $\hat{F}(s)$ IS ANALYTIC.

NOW WE CLAIM,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \hat{F}(s)\, e^{st}\, ds$$



where $\hat{F}(s)$ IS ANALYTIC FOR $RE(s) > \gamma$. HERE, Γ IS CALLED THE BROMWICH CONTOUR.

$\Gamma = \gamma + i\Gamma$

$-\infty < \Gamma < \infty.$

PROOF LET $\hat{G}(k) = \int_{-\infty}^{\infty} g(t) e^{-ikt}\,dt$. THEN $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(k) e^{ikt}\,dk.$

NOW WE LET $g(t) = \begin{cases} e^{-\gamma t} f(t) & t > 0 \\ 0 & t < 0 \end{cases}$

WHERE $|f(t)| \leq c e^{\alpha t}$ AS $t \to \infty$, WHERE $\gamma > \alpha$. HENCE $\int_0^\infty g(t)\,dt < \infty.$

NOW, WE CALCULATE

$$\hat{G}(k) = \int_0^\infty e^{-(\gamma + ik)} f(t)\,dt.$$

LET $s = \gamma + ik$. IT IS THEN DEFINED FOR $RE(s) = \gamma > \alpha$. THE INVERSION FORMULA IS, FOR $t > 0$, THAT

$$2\pi e^{-\gamma t} f(t) = \int_{-\infty}^{\infty} \hat{G}(k) e^{ikt}\,dk.$$

AND THAT $\int_{-\infty}^{\infty} \hat{G}(k) e^{ikt}\,dk = 0$ FOR $t < 0$. THEN, WE LET $s = \gamma + ik$, $ds = i\,dk$.

WE HAVE THAT

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, \hat{G}(s)\,ds$$

$$\hat{G}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

WHERE $\hat{G}(s)$ IS ANALYTIC FOR $RE(s) > \gamma$ ∎.

(ix) $\mathcal{L}(t^n) = \int_0^\infty e^{-st} t^n\,dt$. LET $x = st \to dt = \frac{1}{s}dx \Rightarrow \mathcal{L}(t^n) = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\,dx.$

NOW integration by parts gives $\int_0^\infty e^{-x} x^n\,dx = n!$.

$$\longrightarrow \mathcal{L}(t^n) = n!/s^{n+1}.$$

COMPLEX INTEGRATION

To calculate $f(t)$ we use residue theory. Suppose $F(s)$ has poles in $\mathrm{Re}(s) < \gamma$: then,

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma} F(s)e^{st} = -\lim_{R\to\infty}\int_{C_R}\frac{1}{2\pi i}F(s)e^{st}\,ds + \sum_{j=1}^{N}\mathrm{RES}\left[F(s)e^{st}; s_j\right]$$

Therefore, when $F(s)$ has only poles in region $\mathrm{Re}(s) < \gamma$, we calculate

$$f(t) = \sum_{j=1}^{N}\mathrm{RES}\left[F(s)e^{st}; s_j\right] - \frac{\lim_{R\to\infty}}{2\pi i}\int_{C_R} F(s)e^{st}\,ds.$$

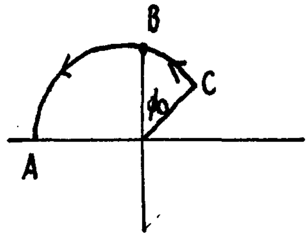
LEMMA Suppose $\exists\, M>0,\ K>0$, such that on C_R $(s = Re^{i\varphi})$,

$$|F(s)| \le M/R^K.$$

Then, we have

$$\lim_{R\to\infty}\left|\int_{C_R} e^{st}F(s)\,ds\right| = 0.$$

PROOF



On CB: we have $s = Re^{i\phi}$ with $\phi_0 < \varphi < \pi/2$.

$$\left|\int_{C_B}\right| = \frac{1}{2\pi}\left|\int_{\phi_0}^{\pi/2} e^{Re^{i\phi}t}F(Re^{i\phi})\,iRe^{i\phi}\,d\phi\right|$$

$$\le \frac{1}{2\pi}\int_{\phi_0}^{\pi/2} R\,e^{Rt\cos\phi}\,|F(Re^{i\phi})|\,d\phi \le \frac{M}{2\pi R^{K-1}}\int_{\phi_0}^{\pi/2} e^{Rt\cos\phi}\,d\phi$$

Now let $\phi = \pi/2 - \varphi$. $\cos\varphi = \cos(\pi/2 - \phi) = \sin\phi$. Then, with $\phi_0 = \sin^{-1}(\gamma/R)$,

$$\left|\int_{C_B}\right| \le \frac{M}{2\pi R^{K-1}}\int_0^{\phi_0} e^{Rt\sin\phi}\,d\phi \le \frac{M}{2\pi R^{K-1}}\int_0^{\phi_0} e^{Rt\sin\phi_0}\,d\phi = \frac{M}{2\pi R^{K-1}}\int_0^{\phi_0} e^{\gamma t}\,dt$$

since $\sin\phi \le \sin\phi_0 = \gamma/R$. Therefore,

$$\left|\int_{C_B}\right| \le \frac{M}{2\pi R^{K-1}}e^{\gamma t}\sin^{-1}\left(\frac{\gamma}{R}\right) \longrightarrow \frac{M\gamma}{2\pi R^{K}}e^{\gamma t} \text{ as } R\to\infty. \Rightarrow \left|\int_{C_B}\right| \to 0,$$

as $R\to\infty$.

Now on AB we calculate

$$\left|\int_{AB}\right| \le \frac{1}{2\pi}\int_{\pi/2}^{\pi}|e^{st}|\,|F(s)|\,|ds| \le \frac{1}{2\pi}\int_{\pi/2}^{\pi} e^{Rt\cos\varphi}|F|\,R\,d\varphi$$

Therefore,

$$\left|\int_{AB}\right| \le \frac{1}{2\pi}\,\frac{M}{R^{K-1}}\int_{\pi/2}^{\pi} e^{tR\cos\varphi}\,d\varphi = \frac{1}{2\pi}\,\frac{M}{R^{K-1}}\int_0^{\pi/2} e^{-Rt\sin\vartheta}\,d\vartheta.\ (\varphi = \pi/2 + \vartheta).$$

This gives upon using $\sin\vartheta \ge 2\vartheta/\pi$ for $0 \le \vartheta \le \pi/2$, that

$$\left|\int_{AB}\right| \le \frac{M}{2\pi R^{K-1}}\int_0^{\pi/2} e^{-2Rt\vartheta/\pi}\,d\vartheta = \frac{M}{R^{K-1}2\pi}\left(\frac{\pi}{2Rt}\right)\left(1-e^{-Rt}\right) \longrightarrow 0$$

as $R\to\infty$, if $t>0$.

EXAMPLE

SUPPOSE THAT $F(s) = \dfrac{P(s)}{Q(s)}$ WHERE $P(s), Q(s)$ POLYNOMIALS WITH $\deg Q > \deg P$.

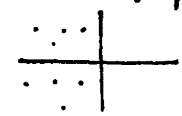
SUPPOSE THAT $Q(s_j) = 0$ $j = 1,..,N$ AND $Q'(s_j) \neq 0$. HENCE s_j IS A SIMPLE ROOT.

NOW WE CALCULATE,

$$f(t) = \mathcal{L}^{-1}[F(s)] = \sum_{j=1}^{N} \operatorname{RES}\left[\frac{P(s)e^{st}}{Q(s)}; s_j\right] = \sum_{j=1}^{N} \frac{P(s_j)}{Q'(s_j)} e^{s_j t}.$$

REMARKS (i) IF ALL $\operatorname{RE}(s_j) < 0 \Rightarrow f(t) \to 0$ AS $t \to \infty$

s- plane



(ii) THE DOMINANT BEHAVIOR FOR LARGE t IS DETERMINED BY THE POLE WITH THE LARGEST REAL PART.

LET $\operatorname{RE}(s_J) > \operatorname{RE}(s_j)$ FOR $j = 1,..,N,\ j \neq J$.

THEN, $$f(t) \approx \frac{P(s_J)}{Q'(s_J)} e^{s_J t} \quad \text{AS } t \to \infty.$$

EXAMPLE

(i) $F(s) = \dfrac{1}{s-a}$, $\Longrightarrow f(t) = e^{at}$.

(ii) $F(s) = \dfrac{\omega}{s^2+\omega^2} \Longrightarrow f(t) = \operatorname{RES}\left[\dfrac{\omega}{s^2+\omega^2} e^{st}; i\omega\right] + \operatorname{RES}\left[\dfrac{\omega}{s^2+\omega^2} e^{st}; -i\omega\right]$

$$f(t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) = \sin(\omega t).$$

(iii) $F(s) = \dfrac{s}{s^2+\omega^2}. \Longrightarrow f(t) = \cos(\omega t)$

(iv) $F(s) = \dfrac{1}{s(s^2+1)} = G(s)H(s)$. $g(t) = 1$, $h(t) = \sin t$. $f(t) = \displaystyle\int_0^t \sin(\tau)\, d\tau = -\cos\tau\Big|_0^t$

THEN, $f(t) = 1 - \cos t$.

(v) $F(s) = \dfrac{1}{(s+1)(s-2)^2}$.

$$f(t) = \operatorname{RES}\left[\frac{e^{st}}{(s+1)(s-2)^2}; -1\right] + \operatorname{RES}\left[\frac{e^{st}}{(s+1)^2(s-2)^2}; 2\right]$$

$$= \frac{e^{-t}}{9} + \lim_{s\to 2}\left[\frac{d}{ds}\left[e^{st}(s-2)^2 F(s)\right]\Big|_{s=2}\right] = \lim_{s\to 2}\left[\frac{d}{ds}\left[(s+1)^{-1}e^{st}\right]\right] + e^{-t}/9.$$

CALCULATING WE GET $f(t) = e^{-t}/9 + \lim\limits_{s\to 2}\left[-(s+1)^{-2}e^{st} + t(s+1)^{-1}e^{st}\right] = te^{2t}/3.$

(vi) GIVEN $y'' - y' - 6y = 1 + e^{-t}$

$y(0) = \alpha,\ y'(0) = 0.$

FIND α SO THAT y IS BOUNDED AS $t \to \infty$.

TAKE LAPLACE TRANSFORMS TO GET

$$(s^2 - s - 6)Y - s\alpha + \alpha = \frac{1}{s} + \frac{1}{s+1}. \longrightarrow Y(s) = \frac{\alpha(s-1) + \frac{1}{s} + \frac{1}{s+1}}{(s-3)(s+2)}.$$

WE OBTAIN,

$$Y(s) = \frac{\alpha s(s^2-1) + 2s+1}{s(s-3)(s+2)(s+1)} = \frac{P(s)}{Q(s)}$$

WE HAVE POLES AT $s = 0, -1, -2, 3$. THE POLE AT $s=3$ GIVES $y \sim (\ )e^{3t}$ AS $t \to \infty$.

HENCE TO HAVE A BOUNDED SOLUTION AS $t \to \infty$, WE REQUIRE

$$y \sim \frac{P(3)}{Q'(3)} e^{3t} + \frac{P(0)}{Q'(0)} e^{0t} + \ldots \quad \text{i.e.} \quad P(3) = 0.$$

THIS GIVES,

$$24\alpha + 7 = 0 \rightarrow \alpha = -7/24.$$

(vii) given $F(s) = \frac{s}{(s^2+1)^2}$, find $f(t)$. WE NOTICE THAT

$$F(s) = -\frac{d}{ds} G(s), \quad \text{WHERE} \quad G(s) = \frac{1}{2}(s^2+1)^{-1}. \rightarrow g(t) = \frac{1}{2}\sin t.$$

RECALLING

$$\mathcal{L}[t\,g(t)] = -G'(s) \longrightarrow \mathcal{L}^{-1}[F(s)] = +\frac{t}{2}\sin t.$$

(viii) LET $F(s) = \frac{1}{(s^2+1)^2}$ pole of order 2 at $s = \pm i$.

$$\text{RES}[\ ; i] = \lim_{s\to i}\left[\frac{d}{ds}\left(\frac{e^{st}}{(s+i)^2}\right)\right] = \lim_{s\to i}\left[\frac{te^{st}}{(s+i)^2} - \frac{2e^{st}}{(s+i)^3}\right] = \frac{-te^{it}}{4} + \frac{e^{it}}{4i}.$$

$$\text{RES}[\ ; -i] = \lim_{s\to -i}\left[\frac{d}{ds}\left(\frac{e^{st}}{(s-i)^2}\right)\right] = \lim_{s\to -i}\left[\frac{te^{st}}{(s-i)^2} - \frac{2e^{st}}{(s-i)^3}\right] = -\frac{te^{-it}}{4} - \frac{1}{4i}e^{-it}.$$

THIS GIVES,

$$f(t) = -\frac{t}{4}\left(e^{it} + e^{-it}\right) + \frac{1}{4i}\left(e^{it} - e^{-it}\right)$$

$$f(t) = -\frac{t}{2}\cos t + \frac{1}{2}\sin t.$$

(ix) LET $X(s) = \frac{1}{s^2(s^2+w^2)}$.

$$F(s) = 1/s^2 \longrightarrow f(t) = t$$

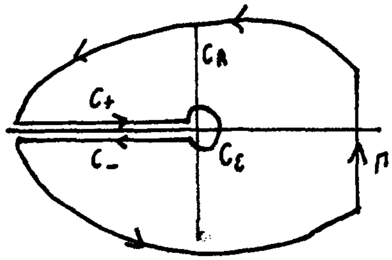
$$G(s) = \frac{1}{s^2+w^2} \longrightarrow g(t) = \frac{1}{w}\sin(wt)$$

THEN $h(t) = \mathcal{L}^{-1}[X(s)] = \int_0^t \frac{\tau}{w}\sin(w(t-\tau))\,d\tau = \frac{t}{w^2} - \frac{\sin(wt)}{w^3}$ integration by parts.

MULTI-VALUED FUNCTIONS

EXAMPLE 1

WE LET $F(s) = s^{-a}$ WITH $0 < a < 1$. WE REQUIRE $F(s)$ ANALYTIC FOR $\mathrm{Re}(s) > \gamma$. HENCE, WE CAN TAKE THE CONTOUR AS SHOWN



SO $f(t) = \frac{1}{2\pi i}\int_\Gamma s^{-a} e^{st}\, ds$.

NOW ON C_ε: $\left|\frac{1}{2\pi i}\int_{C_\varepsilon} s^{-a} e^{st}\, ds\right| \le C\varepsilon^{1-a} \to 0$ AS $\varepsilon \to 0$.

NOW $\left|\int_{C_R}\right| \to 0$ AS $R \to \infty$, SINCE $|F| \to 0$ AS $R \to \infty$.

NOW WE HAVE, $\frac{1}{2\pi i}\int_\Gamma + \frac{1}{2\pi i}\left(\int_{C_+} + \int_{C_-}\right) = 0.$

THEREFORE, $$f(t) = -\frac{1}{2\pi i}\int_{C_+} - \frac{1}{2\pi i}\int_{C_-}.$$

NOW ON C_+: $s = re^{i\pi}$ SO $ds = -dr$. $\longrightarrow \frac{1}{2\pi i}\int_{C_+} = \frac{1}{2\pi i}\int_\infty^0 e^{-rt} r^{-a} e^{-i\pi a}(-dr)$

NOW ON C_-: $s = re^{-i\pi}$ SO $ds = -dr$ $\longrightarrow \frac{1}{2\pi i}\int_{C_-} = \frac{1}{2\pi i}\int_0^\infty e^{-rt} r^{-a} e^{i\pi a}(-dr)$.

THEREFORE, $$f(t) = \frac{-1}{2\pi i}\int_0^\infty r^{-a}\left(e^{-i\pi a} - e^{i\pi a}\right)e^{-rt}\, dr = \frac{1}{\pi}\int_0^\infty r^{-a}\sin(\pi a)\, e^{-rt}\, dr$$

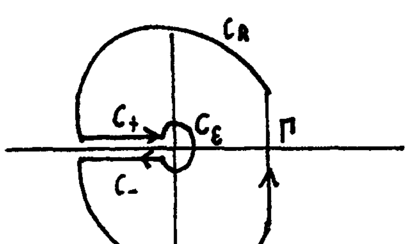
NOW $$f(t) = \frac{\sin(\pi a)}{\pi}\int_0^\infty r^{-a} e^{-rt}\, dr.$$ LET $x = rt$, $dr = \frac{1}{t}dx$, $r^{-a} = x^{-a}t^{a}$.

HENCE, $$f(t) = t^{a-1}\frac{\sin(\pi a)}{\pi}\int_0^\infty x^{-a} e^{-x}\, dx.$$

RECALL $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, dx$ is the gamma function. Therefore,
$$f(t) = t^{a-1}\frac{\sin(\pi a)}{\pi}\Gamma(1-a).$$

EXAMPLE 2 FIND THE INVERSE OF $F(s) = e^{-a\sqrt{s}}/s$ WHERE $a \ge 0$.

WE TAKE THE CONTOUR AS SHOWN:



WE HAVE $$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\frac{1}{s}e^{-a\sqrt{s}+st}\, ds.$$

NOW $$f(t) = -\frac{1}{2\pi i}\int_{C_+} - \frac{1}{2\pi i}\int_{C_-} - \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{C_\varepsilon}.$$

NOW WE CALCULATE

$$\lim_{\varepsilon \to 0} \left[-\frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{e^{-a\sqrt{s}}}{s} e^{st} \, ds \right] \longrightarrow -\frac{1}{2\pi i} \oint_{C_\varepsilon} \frac{1}{s} \, ds = 1.$$

NOW ON C_+: $s = re^{i\pi}$ $\quad s^{1/2} = r^{1/2} e^{i\pi/2} = i r^{1/2}$.

C_-: $s = re^{-i\pi}$ $\quad s^{1/2} = r^{1/2} e^{-i\pi/2} = -i r^{1/2}$.

THUS

$$f(t) = 1 - \frac{1}{2\pi i} \int_\infty^0 \frac{e^{-rt} e^{-iar^{1/2}}}{-r} (-dr) - \frac{1}{2\pi i} \int_0^\infty \frac{e^{-rt + iar^{1/2}}}{-r} (-dr).$$

HENCE,

$$f(t) = 1 - \frac{1}{2\pi i} \int_0^\infty \frac{e^{-rt}}{r} \left(e^{iar^{1/2}} - e^{-iar^{1/2}} \right) dr$$

WE CALCULATE,

$$f(t) = 1 - \frac{1}{\pi} \int_0^\infty \frac{e^{-rt}}{r} \sin(ar^{1/2}) \, dr$$

LET $u = r^{1/2}$, $r = u^2$, $dr = 2u\,du$. THUS

$$f(t) = 1 - \frac{2}{\pi} \int_0^\infty e^{-u^2 t} \frac{\sin(au)}{u} \, du.$$

WE DEFINE

$$I(a) = \frac{2}{\pi} \int_0^\infty e^{-u^2 t} \frac{\sin(au)}{u} \, du.$$

WE GET,

$$I'(a) = \frac{2}{\pi} \int_0^\infty e^{-u^2 t} \cos(au) \, du = \frac{2}{\pi} \, \mathrm{RE}\left[\int_0^\infty e^{-u^2 t + iau} \, du \right]$$

$$I'(a) = \mathrm{RE}\left(\frac{1}{\pi} \int_{-\infty}^\infty e^{-t(u^2 - iau/t - a^2/4t^2) - a^2/4t} \, du \right) = \frac{e^{-a^2/4t}}{\pi} \, \mathrm{RE}\left[\int_{-\infty}^\infty e^{-t\left(u - \frac{ia}{2t}\right)^2} du \right.$$

NOW LET $2\sigma^2 = 1/t$, SO THAT $\sigma = (1/2t)^{1/2}$. HENCE,

$$I'(a) = \frac{1}{\pi} e^{-a^2/4t} \sqrt{2\pi}\, \sigma = \frac{1}{\sqrt{\pi t}} e^{-a^2/4t}.$$

WE CALCULATE, $I(a) = \dfrac{1}{\sqrt{\pi t}} \displaystyle\int_0^a e^{-\lambda^2/4t} \, d\lambda$. LET $\varsigma = \lambda/2t^{1/2}$. $d\lambda = 2t^{1/2} d\varsigma$.

THUS $I(a) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{a/2t^{1/2}} e^{-\varsigma^2} d\varsigma = \mathrm{ERF}\left(a/2t^{1/2} \right)$, WHERE $\mathrm{ERF}(z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^z e^{-\varsigma^2} d\varsigma$.

FINALLY,

$$f(t) = 1 - \mathrm{ERF}\left(a/2\sqrt{t} \right).$$

REMARKS

1. $\mathrm{ERF}(Z) \equiv \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^Z e^{-\xi^2}\,d\xi$. NOW $\mathrm{ERF}(0) = 0$, $\mathrm{ERF}(\infty) = 1$

NOW $\mathrm{ERFC}(Z) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_Z^\infty e^{-\xi^2}\,d\xi$, $\mathrm{ERFC}(\infty) = 0$, $\mathrm{ERFC}(0) = 1$.

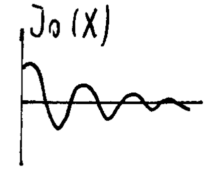
ALSO $\mathrm{ERF}(Z) + \mathrm{ERFC}(Z) = 1$.

2. THEREFORE IF $F(s) = \dfrac{e^{-a\sqrt{s}}}{s}$

$$f(t) = 1 - \mathrm{ERF}\left(a/2\sqrt{t}\right) = \mathrm{ERFC}\left(a/2\sqrt{t}\right).$$

PROBLEM CONSIDER $xy'' + y' + xy = 0 \quad X \geq 0$

$y(0) = 1$, $y'(0)$ FINITE



THE SOLUTION IS $y = J_0(X)$ WHERE $J_0(X)$ IS THE BESSEL FUNCTION OF ORDER ZERO OF FIRST KIND. WE WILL FIND AN INTEGRAL REPRESENTATION OF $J_0(X)$ USING A LAPLACE TRANSFORM SOLUTION.

$$\mathcal{L}(xy'') + \mathcal{L}(y') + \mathcal{L}(xy) = 0$$

BUT $\mathcal{L}(xy'') = \displaystyle\int_0^\infty x y'' e^{-sx}\,dX$

SO $\mathcal{L}(xy'') = -\dfrac{d}{ds}\displaystyle\int_0^\infty y'' e^{-sx}\,dx$

$$= -\frac{d}{ds}\left[s^2 \bar{Y}(s) - s y(0) - y'(0)\right]$$

$$\rightarrow -\frac{d}{ds}\left[s^2 \bar{Y} - s y(0) - y'(0)\right] + s\bar{Y} - y(0) - \bar{Y}' = 0$$

THIS YIELDS THAT

$$\frac{d}{ds}(s^2 \bar{Y}) + \bar{Y}' - s\bar{Y} = 0$$

ALSO $\mathcal{L}(xy) = \displaystyle\int_0^\infty x y e^{-sx}\,dx = -\dfrac{d}{ds}\displaystyle\int_0^\infty y e^{-sx}\,dx$

OR EQUIVALENTLY $(s^2+1)\bar{Y}' + s\bar{Y} = 0 \longrightarrow \dfrac{\bar{Y}'}{\bar{Y}} = -\dfrac{s}{s^2+1} \rightarrow (\ln Y)' = -\dfrac{1}{2}\log(s^2+1)'$

HENCE $\bar{Y}(s) = \dfrac{C}{\sqrt{s^2+1}}$, where C IS ARBITRARY AT THIS STAGE.

NOW $y(0) = 1$ $\quad X \to 0$ CORRESPONDS TO $S \to \infty$.

SO $\dfrac{C}{(s^2+1)^{1/2}} = \displaystyle\int_0^\infty y(x) e^{-sx}\,dx$

HENCE FOR $s \gg 1$ $\quad \dfrac{C}{s} \sim \displaystyle\int_0^\infty e^{-sx}\,dx = \dfrac{1}{s}$.

THIS GIVES $C = 1$, WHICH IMPLIES $J_0(x) = \dfrac{1}{2\pi i}\displaystyle\int_{a-i\infty}^{a+i\infty} \dfrac{e^{sx}}{\sqrt{s^2+1}}\,ds$. $(a > 0)$

LAPLACE TRANSFORM AND PDE'S

PROBLEM 1 CONSIDER $u_t = D u_{xx} \quad 0 < x < \infty, \; t > 0$

$$u(x,0) = 0, \quad u(0,t) = u_0, \quad u_0 \text{ CONSTANT.}$$

NOW TAKE LAPLACE TRANSFORMS $U(x,s) = \mathcal{L}(u)$. NOW WE CALCULATE,

$$\mathcal{L}(u_t) = D\,\mathcal{L}(u_{xx}), \quad \text{WITH} \quad U(x,s) = \mathcal{L}(u(x,t)).$$

WE CALCULATE,

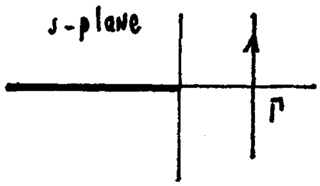
$$sU = D U_{xx} \longrightarrow U = a_0 e^{-\sqrt{s}\,x/\sqrt{D}} + b_0 e^{\sqrt{s}\,x/\sqrt{D}}$$

$$U(0,s) = u_0/s$$

WE CALCULATE $U(x,s) = \dfrac{u_0}{s} e^{-\sqrt{s/D}\,x}$

NOW WE NEED $\mathrm{RE}(\sqrt{s}) > 0$. $\sqrt{s} = \sqrt{R}\, e^{i\varphi/2} \longrightarrow \cos(\varphi/2) > 0 \rightarrow -\pi < \varphi < \pi$.

THUS, WE TAKE BRANCH CUT ON NEGATIVE REAL AXIS.



$$u(x,t) = \frac{u_0}{2\pi i} \int_{\Gamma} \frac{1}{s} e^{-\sqrt{s/D}\,x} e^{st}\, ds$$

WE CALCULATED THIS EARLIER, WITH THE RESULT

$$u(x,t) = u_0 \,\mathrm{ERFC}\left(x/2\sqrt{Dt}\right).$$

PROBLEM 2 WE WANT TO SOLVE

$$u_t = D u_{xx} \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = \delta(x - x_0)$$

WE CALCULATE THE LAPLACE TRANSFORM, $U(x,s) = \int_0^\infty e^{-st} u(x,t)\, dt$

$$sU - \delta(x - x_0) = D U_{xx}$$

WE HAVE

$$D U_{xx} - sU = -\delta(x - x_0) \quad -\infty < x < \infty$$

$$U \text{ BOUNDED AS } |x| \to \infty.$$

THE CONTINUITY AND JUMP CONDITIONS ARE $[U]\big|_{x_0} = 0, \quad [D U_x]\big|_{x = x_0} = -1$.

NOW WE GET

$$U(x,s) = \begin{cases} A e^{+\sqrt{s/D}\,(x - x_0)} & x < x_0 \\ A e^{-\sqrt{s/D}\,(x - x_0)} & x > x_0. \end{cases}$$

NOW $D U_x\big|_{x_0^+} - D U_x\big|_{x_0^-} = -1 \longrightarrow -2\sqrt{\frac{s}{D}}\, A = -\frac{1}{D} \longrightarrow A = \dfrac{1}{2\sqrt{Ds}}$.

NOW WE CALCULATE,

$$\bar{U}(X,S) = \frac{1}{2\sqrt{DS}}\, e^{-\sqrt{S/D}\,|X-X_0|}$$

NOW RECALL, $\mathcal{L}^{-1}\left[\frac{e^{-a\sqrt{S}}}{\sqrt{S}}\right] = \frac{1}{\sqrt{\pi t}}\, e^{-a^2/4t}$, WHERE $a = |X-X_0|/\sqrt{D}$.

HENCE, $$U(X,t) = \frac{1}{2\sqrt{\pi D t}}\, e^{-(X-X_0)^2/4Dt}.$$

<u>EXAMPLE 3</u> NOW CONSIDER

$$U_t = D U_{XX} \qquad 0 < X < 1$$
$$U_X(0,t) = 0, \quad U(1,t) = 1, \quad U(X,0).$$

NOW WE TAKE LAPLACE TRANSFORMS TO GET

$$D\bar{U}_{XX} - S\bar{U} = 0 \qquad 0 < X < 1$$
$$\bar{U}_X = 0 \text{ ON } X = 0, \qquad \bar{U} = \frac{1}{S} \text{ ON } X = 1.$$

WE CALCULATE,

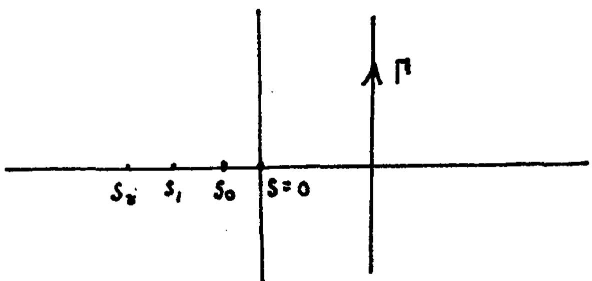
$$\bar{U}(X,S) = \frac{1}{S}\,\frac{\cosh\left(\sqrt{S/D}\,X\right)}{\cosh\left(\sqrt{S/D}\right)}.$$ $\bar{U}$ has only poles, no branch cuts.

NOW $S=0$ IS A SIMPLE POLE. ALSO POLES OF $\cosh\left(\sqrt{S/D}\right) = 0$. HENCE

$$\sqrt{S/D} = \left(K+\tfrac{1}{2}\right)\pi i, \quad \text{THEN} \quad S_K = -D\left(K+\tfrac{1}{2}\right)^2\pi^2 \qquad K = 0,1,2,\ldots$$

NOTICE $S_0 = S_{-1}$ $S_1 = S_{-2}$.

NOW WE USE RESIDUE THEOREM



$$U(X,t) = \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{S}\,\frac{\cosh\left(\sqrt{S/D}\,X\right)}{\cosh\left(\sqrt{S/D}\right)}\, e^{St}\, dS$$

$$U(X,t) = \operatorname{RES}\left[\ \ ; S=0\right] + \sum_{K=0}^{\infty} \operatorname{RES}\left[\frac{1}{S}\,\frac{\cosh\left(\sqrt{S/D}\,X\right)}{\cosh\left(\sqrt{S/D}\right)}\, e^{St}; S_K\right]$$

$$U(X,t) = 1 + \sum_{K=0}^{\infty} \frac{e^{S_K t}\cosh\left[\left(K+\frac{1}{2}\right)\pi i X\right]}{S_K\,\frac{1}{2}\,\frac{1}{(S_K D)^{1/2}}\,\sinh\left[\left(K+\frac{1}{2}\right)\pi i\right]} = 1 + 2\sum_{K=0}^{\infty} \frac{\cos\left(\left(K+\frac{1}{2}\right)\pi X\right) e^{S_K t}}{\left(K+\frac{1}{2}\right)\pi i \quad i(-1)^K}$$

THEREFORE, $$U = 1 + \frac{2}{\pi}\sum_{K=0}^{\infty} \frac{(-1)^{K+1}}{\left(K+1/2\right)}\cos\left(\left(K+\tfrac{1}{2}\right)\pi X\right) e^{-D\left(K+1/2\right)^2\pi^2 t}$$

NOW FOR $t >> 1$ WE GET $u(x,t) \sim 1 - \frac{4}{\pi} e^{-D^2\pi^2 t/4} \cos(\pi x/2)$.

NOW LARGE s CORRESPONDS TO SMALL t. FOR $s >> 1$ WE USE $\cosh(z) \sim e^z/2$, WE OBTAIN

$$U(x,s) \sim \frac{1}{s} e^{-\sqrt{s/D}(1-x)}$$

LETTING $a = (1-x)/\sqrt{D}$ AND RECALLING $\mathcal{L}^{-1}\left[\frac{1}{s} e^{-a\sqrt{s}}\right] = \text{erfc}(a/2\sqrt{t})$.

THUS, $$u(x,t) \sim \text{erfc}\left((1-x)/2\sqrt{Dt}\right) \quad \text{FOR } t << 1.$$

<u>EXAMPLE</u> (WAVE EQUATION)

CONSIDER
$$u_{tt} = u_{xx} \quad 0 < x < L,\ t > 0$$
$$u(x,0) = u_t(x,0) = 0$$
$$0 = u(0,t);\ u(L,t) = 1.$$

WE HAVE TWO REPRESENTATIONS OF THE SOLUTION:

<u>METHOD 1</u> WE TAKE LAPLACE TRANSFORMS TO OBTAIN,

$$U_{xx} - s^2 U = 0 \quad 0 < x < L$$
$$U(0,s) = 0,\quad U(L,s) = \frac{1}{s}.$$

WE SOLVE TO GET $U(x,s) = a_0 \cosh(sx) + b_0 \sinh(sx)$. SATISFYING THE BOUNDARY CONDITION WE OBTAIN

$$U(x,s) = \frac{1}{s} \frac{\sinh(sx)}{\sinh(sL)}.$$

THE POLES ARE WHERE $sL = n\pi i$ WITH $n = \pm 1, \pm 2, \ldots$ ALSO $n = 0$.

THEREFORE,

$$u(x,t) = \text{RES}\left[\frac{1}{s}\frac{\sinh(sx)}{\sinh(sL)} e^{st}; 0\right] + \sum_{n=1}^{\infty} \text{RES}\left[\frac{1}{s}\frac{\sinh(sx)}{\sinh(sL)} e^{st}; \frac{n\pi i}{L}\right] + \sum_{n=-1}^{-\infty} \text{RES}\left[\frac{1}{s}\frac{\sinh(sx)}{\sinh(sL)} e^{st}; \frac{n\pi i}{L}\right]$$

WE NOW CALCULATE

$$u(x,t) = x/L + \sum_{n=1}^{\infty} \frac{L}{n\pi i} \frac{\sinh(n\pi i x/L)}{L\cosh(n\pi i)} e^{n\pi i t/L} + \sum_{n=-1}^{-\infty} \frac{L}{n\pi i} \frac{\sinh(n\pi i x/L)}{L\cosh(n\pi i)} e^{n\pi i t/L}$$

NOW $\cosh(n\pi i) = (-1)^n$. WE OBTAIN,

$$u(x,t) = x/L + \sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin(n\pi x/L)\, e^{n\pi i t/L} + \sum_{n=-1}^{-\infty} \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{n\pi i t/L}.$$

↑ Replace n by $-n'$.

THEN, WE CALCULATE

$$u(x,t) = x/L + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right).$$

<u>METHOD 2</u>

$$U(x,s) = \frac{1}{s}\frac{\sinh(sx)}{\sinh(sL)} = \frac{1}{s}\left[\frac{e^{sx}-e^{-sx}}{e^{sL}-e^{-sL}}\right] = \frac{1}{s}\frac{e^{sx}}{e^{sL}}\left(\frac{1-e^{-2sx}}{1-e^{-2sL}}\right)$$

NOW USING $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ FOR $|x|<1$, WE CALCULATE

$$U(x,s) = \frac{1}{s} e^{-s(L-x)}\left[1 - e^{-2sx}\right]\left[1 + e^{-2sL} + e^{-4sL} + \dots + e^{-2nsL} + \dots\right]$$

THIS GIVES,

$$U(x,s) = \frac{1}{s}\left[e^{-s(L-x)} - e^{-s(L+x)}\right]\left[1 + e^{-2sL} + e^{-4sL} + \dots + e^{-2nsL} + \dots\right]$$

$$U(x,s) = \frac{1}{s}\left[e^{-s(L-x)} + e^{-s(3L-x)} + \dots + e^{-s((2n+1)L-x)} + \dots\right]$$

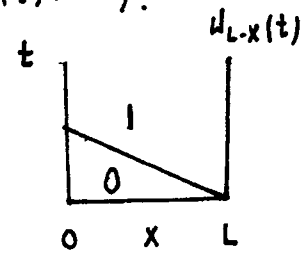
$$- \frac{1}{s}\left[e^{-s(L+x)} + e^{-s(3L+x)} + \dots + e^{-s((2n+1)L+x)}\right]$$

NOW RECALL $\mathcal{L}(u_a(t)) = \frac{1}{s}e^{-as}$ WITH $u_a(t) = \begin{cases} 1 & \text{IF } t > a \\ 0 & \text{IF } 0 < t < a. \end{cases}$

HENCE, $u(x,t) = \left(u_{L-x}(t) + u_{3L-x}(t) + \dots + u_{(2n+1)L-x}(t) + \dots\right)$

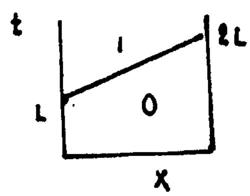
$$- \left(u_{L+x}(t) + u_{3L+x}(t) + \dots + u_{(2n+1)L+x}(t) + \dots\right).$$

(Reflected waves)

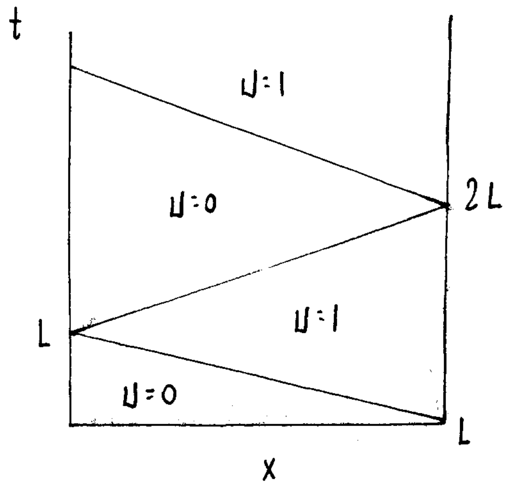


<u>REMARK</u> $u_{L-x}(t) = \begin{cases} 1 & \text{IF } t > L-x \rightarrow x > L-t \\ 0 & \text{IF } 0 < t < L-x \end{cases}$

$u_{L+x}(t) = \begin{cases} 1 & \text{IF } t > L+x \\ 0 & \text{IF } t < L+x \end{cases}$



THEREFORE THE SOLUTION HAS THE FORM $u = u_{L-x}(t) - u_{L+x}(t) + \dots$



EXAMPLE FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION TO THE WAVE EQUATION $u_{tt} = u_{xx}$

$$u(x,0) = u_t(x,0) = 0, \quad u(0,t) = 0, \quad u(L,t) = f(t)$$

THIS CORRESPONDS TO BOUNDARY FORCING WHICH GENERATES WAVES IN $0 < x < L$.

SOLUTION WE TAKE $U(x,s) = \int_0^\infty e^{-st} u(x,t)\,dt$. NOW WITH $u = u_t = 0$ AT $t = 0$

THIS GIVES $$s^2 U = U_{xx}$$
$$U(0,s) = 0, \quad U(L,s) = F(s) \quad \text{WHERE} \quad F(s) = \mathcal{L}(f(t)).$$

THE SOLUTION IS $U(x,s) = A\sinh(sx)$ SO $F(s) = A\sinh(sL)$,

WHICH YIELDS $$U(x,s) = F(s)\frac{\sinh(sx)}{\sinh(sL)}$$

WE WRITE THIS AS $U(x,s) = F(s)\,G(s)$ $\quad G(s) = s\,\Phi(x,s)$

WHERE $\Phi(x,s) = \frac{1}{s}\frac{\sinh(sx)}{\sinh(sL)}$. THE INVERSE TRANSFORM $\phi(x,t) = \mathcal{L}^{-1}(\Phi(x,s))$

WAS FOUND IN PREVIOUS PROBLEM. HENCE $\mathcal{L}^{-1}[s\,\Phi(x,s)] = \frac{d}{dt}\phi(x,t)$ SINCE $\phi(x,0) = 0$

THEN USING CONVOLUTION THEOREM WITH $\mathcal{L}^{-1}[F(s)] = f(s)$, $\mathcal{L}^{-1}[G(s)] = \frac{d}{dt}\phi(x,t)$

WE OBTAIN $$u(x,t) = \int_0^t \left(\frac{d}{d\tau}\Phi(x,\tau)\right) f(t-\tau)\,d\tau.$$

DIFFUSION IN SEMI-INFINITE MEDIUM

CONSIDER $u_t = D u_{xx} \quad 0<x<\infty, \ t>0$

$$u(0,t) = h(t), \quad u(x,0) = 0, \quad u \to 0 \text{ as } x \to \infty \text{ FIXED } t.$$

NOW TAKE LAPLACE TRANSFORMS

$$s\bar{U} = D\bar{U}_{xx} \qquad \text{SO} \qquad \bar{U}(x,s) = Ae^{-\sqrt{s/D}\,x} + Be^{\sqrt{s/D}\,x}$$

$$\bar{U}(0,s) = H(s)$$

WE SET $B = 0$ SO THAT $\bar{U} \to 0$ AS $x \to \infty$.

THEN $\bar{U}(0,s) = H(s)$ GIVES

$$\bar{U}(x,s) = H(s)e^{-\sqrt{s/D}\,x}$$

NOW $$\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right) = \frac{a}{2\sqrt{\pi}\,t^{3/2}}\,e^{-a^2/4t}$$

SO THAT $$\mathcal{L}^{-1}\left(e^{-\sqrt{s/D}\,x}\right) = \frac{x}{2\sqrt{\pi D}\,t^{3/2}}\,e^{-x^2/4Dt}$$

$$\mathcal{L}^{-1}(H(s)) = h(t)$$

AND THE CONVOLUTION THEOREM GIVES

$$u(x,t) = \int_0^t h(t-\tau)\,\frac{x}{2\sqrt{\pi D}\,\tau^{3/2}}\,e^{-x^2/4D\tau}\,d\tau$$

DELAY-DIFFERENTIAL EQUATIONS

CONSIDER $\quad y'(t) = -a\,y(t-1) \quad 0 \le t \le \infty$

WITH $\quad y(t) \equiv 1 \quad$ FOR $\quad -1 \le t \le 0.$

THIS IS A DELAY EQUATION. WHAT IS THE BEHAVIOR AS $t \to \infty$ FOR DIFFERENT VALUES OF a.

THEN $\quad Y(s) = \mathcal{L}(y(t))$

$$sY(s) - y(0) = -a\int_0^\infty y(t-1)e^{-st}\,dt = -a\int_0^1 y(t-1)e^{-st}\,dt - a\int_1^\infty y(t-1)e^{-st}\,dt$$

SO $\quad sY(s) - 1 = -a\int_{-1}^{0} y(\xi)e^{-s(\xi+1)}\,d\xi - a\left(\int_0^\infty y(\xi)e^{-s\xi}\right)e^{-s}$

THIS GIVES, $\quad sY(s) - 1 = -a\left(\int_{-1}^{0} e^{-s\xi}\,d\xi\right)e^{-s} - a e^{-s}Y(s)$

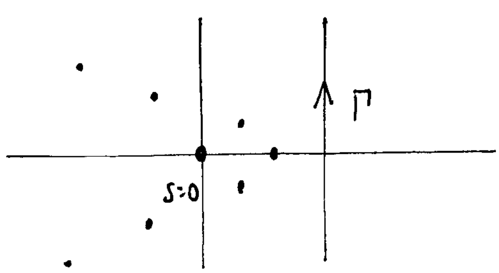
$$= -a e^{-s}\left(-\frac{1}{s}e^{-s\xi}\Big|_{-1}^{0}\right) - a e^{-s}Y(s)$$

SO $\quad (s + ae^{-s})Y(s) = 1 + \dfrac{a}{s}e^{-s}\left(1 - e^{s}\right) = 1 + \dfrac{a}{s}e^{-s} - \dfrac{a}{s}$

THIS GIVES $\quad Y(s) = \dfrac{s - a + ae^{-s}}{s(s + ae^{-s})}$

NOTICE THAT THE INVERSE TRANSFORM IS $\quad y(t) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma Y(s)e^{st}\,ds$

WITH ALL ZEROES OF $s(s + ae^{-s}) = 0$ TO THE LEFT OF VERTICAL LINE Γ.



THERE IS A POLE AT $S = 0$ AND AT THE ZEROES OF $p(S) = S + ae^{-S} = 0$.

<u>REMARK</u> IF WE INITIALLY SUBSTITUTED $y = e^{\lambda t}$ WE WOULD OBTAIN $\quad \lambda = -a e^{-\lambda}$

SO THAT λ IS A ROOT OF $\quad p(\lambda) = \lambda + ae^{-\lambda} = 0$

FOR STABILITY WE NEED $\operatorname{Re}\lambda < 0$.

EXAMPLE SUPPOSE $a=-1$. FIND BEHAVIOR AS $t\to\infty$.

WHEN $a=-1$ WE HAVE $p(s)=s-e^{-s}=0$. WE NEED THE POLE WITH THE LARGEST REAL PART.

WE LET $s=x+iy$ SO THAT $x+iy=e^{-x}(\cos y - i\sin y)$

SO $x=e^{-x}\cos y \qquad y=-e^{-x}\sin y$

NOW THE POLE WITH LARGEST REAL PART x HAS xe^{+x} MAXIMIZED. THIS MEANS THAT $y=0$ AND $xe^{x}=1 \Rightarrow x_*=.567$ BY NEWTON'S METHOD.

HENCE LET $S_* = X_*$ SO THAT

$$y(t)=\frac{1}{2\pi i}\int_{\Gamma} Y(s)e^{st}\,ds \sim \mathrm{RES}\left[\frac{s+1-e^{-s}}{s(s+ae^{-s})}\,;\,S_*\right]e^{S_* t} \quad \text{AS } t\to\infty.$$

HENCE $$y(t)\sim\left(\frac{S_*+1-e^{-S_*}}{S_*\left(1+e^{-S_*}\right)}\right)e^{S_* t}=\frac{e^{S_* t}}{S_*(1+e^{-S_*})}$$ SINCE $S_*-e^{-S_*}=$

SO $$y(t)\sim\frac{e^{S_* t}}{S_*(1+e^{-S_*})}$$ AS $t\to\infty$ WHERE $S_*=.567$

(THIS GIVES $y(t)\sim 1.125\,e^{.567t}$ AS $t\to\infty$)

EXAMPLE (OSCILLATIONS)

NOW SUPPOSE $a>0$. WHAT CAN WE SAY ABOUT THE ZEROES OF

$$p(s)\equiv s+ae^{-s}=0.$$

UNDER WHAT CONDITIONS ARE $\mathrm{RE}(s)\lessgtr 0$?

SUPPOSE WE LOOK FOR s ON IMAGINARY AXIS. WE SET $s=iy$ AND LET $y>0$ (WLOG SINCE $s=-iy$ WOULD ALSO WORK).

THEN $p(iy)=iy+ae^{iy}=iy+a(\cos y - i\sin y)=0$

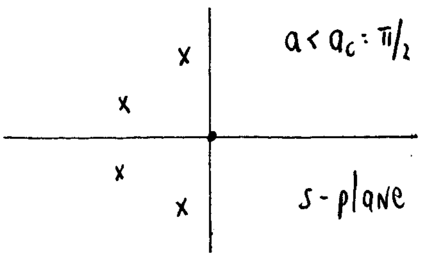
$p(iy) = a\cos y + i(y \mp a\sin y)$

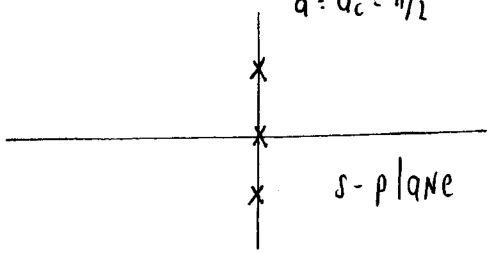
NOTICE $\quad \text{RE } p = 0 \quad$ WHEN $\quad y = \pi/2, 3\pi/2, \ldots$

$\text{IM } p = 0 \quad$ WHEN $\quad y \mp a\sin y = 0 \;\rightarrow\; a = \frac{\pi}{2}$ WHEN $y = \frac{\pi}{2}$.

THEREFORE, WHEN $a = \pi/2$ WE HAVE $s = \pm i\pi/2$ ROOTS ON IMAGINARY AXIS.

WE WOULD LIKE TO PROVE THAT IF $0 < a < a_c = \pi/2$ WE HAVE $\text{RE } s < 0$ WHEREAS FOR $a = a_c = \pi/2 \rightarrow s = \pm i\pi/2$. HENCE WE CAN GET OSCILLATIONS FROM

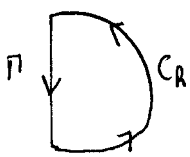




$$y'(t) = -\frac{\pi}{2}\, y(t-1)$$

AS $t \to \infty$.

NOW TO PROVE THIS WE USE THE WINDING NUMBER CRITERION



$$N_0(p) = \frac{1}{2\pi}\left[\Delta_\Gamma \arg(p(iy)) + \lim_{R\to\infty} \Delta_{C_R} \arg(p(s))\right]$$

$N_0(p) = \#$ ZEROES OF p IN $\text{RE}(s) > 0$.

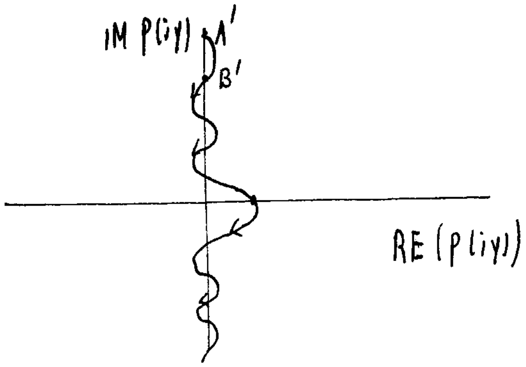
NOW $\quad p(s) = s\left(1 + \frac{ae^{-s}}{s}\right)$ WITH $s = Re^{i\alpha}$ SO $\lim_{R\to\infty} \Delta_{C_R} \arg(p(s)) = \pi$.

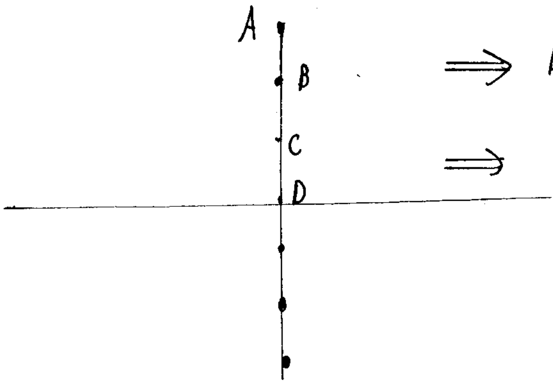
HENCE $\quad N_0(p) = \frac{1}{2\pi}\left[\pi + \Delta_\Gamma \arg(p(s))\right]$

NOW ON Γ, $s = iy$ SO $\quad p(iy) = a\cos y + i(y - a\sin y)$.

SUPPOSE $0 < a < a_c = \pi/2$. THEN IF $\cos y = 0$ WE HAVE $\text{IM}[p(iy)] = y - a\sin y > 0$

THUS $\text{RE}[p(iy)] = 0 \rightarrow \text{IM}[p(iy)] > 0$ FOR $y > 0$.





$\Rightarrow \Delta_\Gamma \arg(p(iy)) = -\pi$

$\Rightarrow N_0(p) = 0$

when $0 < a < a_c = \frac{\pi}{2}$

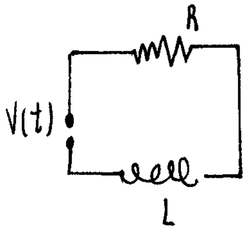
LAPLACE TRANSFORMS AND PERIODIC SOLUTIONS

SUPPOSE THAT f IS PERIODIC WITH $f(t) = f(t+T)$.

THEN WE SHOWED EARLIER THAT

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

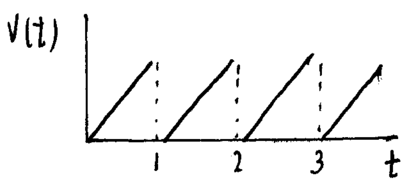
AS AN EXAMPLE OF THE USE OF THIS CONSIDER AN RL CIRCUIT WITH CURRENT $I(t)$ AS SHOWN



THEN $L\dfrac{dI}{dt} + RI = V(t)$ $\quad I(0) = 0.$

$L, R > 0$ CONSTANTS

SUPPOSE THAT $V(t)$ IS A PERIODIC SAWTOOTH FUNCTION AS SHOWN



$V(t+1) = V(t)$ FOR $t > 1$

$V(t) = t$ FOR $0 < t < 1$

NOW LET $J(s) = \mathcal{L}(I(t))$ WE WANT TO DETERMINE $J(s)$ AND THEN RECOVER $I(t)$. WE WRITE

$$\mathcal{L}\left(L\frac{dI}{dt}\right) + R\,\mathcal{L}(I) = \mathcal{L}(V(t)) = V(s)$$

$$L[sJ - I(0)] + RJ = V(s) = \frac{\int_0^1 te^{-st}\,dt}{1-e^{-s}} = \frac{1}{1-e^{-s}}\left[\left(-\frac{1}{s^2}-\frac{t}{s}\right)e^{-st}\Big|_0^1\right]$$

HENCE $$J(sL+R) = \frac{1}{1-e^{-s}}\left(\left(-\frac{1}{s^2}-\frac{1}{s}\right)e^{-s} + \frac{1}{s^2}\right) = \frac{1}{(1-e^{-s})}\left(+\frac{1}{s^2}(1-e^{-s}) - \frac{1}{s}e^{-s}\right)$$

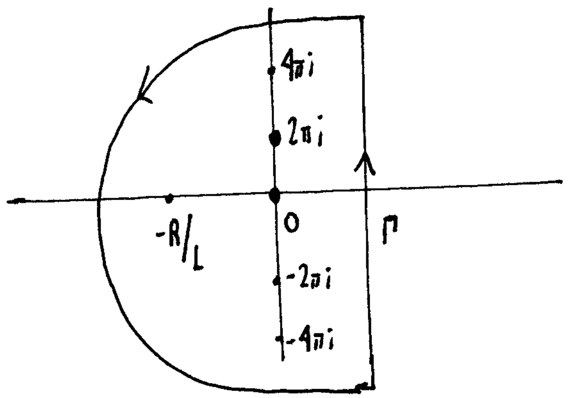
SO $$J(sL+R) = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})} = \frac{1}{s^2} - \frac{1}{s(e^{s}-1)} = \frac{1}{s^2} + \frac{1}{s(1-e^{s})}$$

SO $$\mathcal{L}(I(t)) = J(s) = \left(\frac{1}{s^2} + \frac{1}{s(1-e^{+s})}\right)\frac{1}{sL+R}$$

NOTICE THAT WE HAVE A SIMPLE POLE AT $S = -R/L < 0$,

A DOUBLE POLE AT $S = 0$ AND SIMPLE POLES WHEN $1 - e^{S} = 0$ WHICH

GIVES $S = 2K\pi i \quad K = \pm 1, \pm 2, \ldots$



SO $$I(t) = \frac{1}{2\pi i} \int_{\Gamma} J(s) e^{st}\, ds$$

$$(\ast) \qquad = \mathrm{RES}\left[J e^{st} ; 0\right] + \mathrm{RES}\left[J e^{st} ; -R/L\right] + \sum_{\substack{K=-\infty \\ K \neq 0}}^{\infty} \mathrm{RES}\left[e^{st} J ; 2K\pi i\right]$$

(i) NOW FOR $S = -R/L$ WE OBTAIN BY P/Q RESIDUE EVALUATION METHOD

THAT $$\mathrm{RES}\left[J e^{st} ; -R/L\right] = \frac{e^{-Rt/L}}{L}\left(\frac{1}{s^2} + \frac{1}{s(1-e^{s})}\right)\Bigg|_{s=-R/L} = \frac{e^{-Rt/L}}{L}\left(\frac{L^2}{R^2} - \frac{L}{R(1-e^{-R/L})}\right)$$

SO $$\mathrm{RES}\left[J e^{st} ; -R/L\right] = e^{-Rt/L}\left(\frac{L}{R^2} - \frac{1}{R(1-e^{-R/L})}\right)$$

(ii) FOR $S = 0$ WE HAVE A "DOUBLE" POLE. WE CALCULATE THE a_{-1} TERM FROM LAURENT SERIES

$$\mathrm{RES}\left[J(s) e^{st} ; 0\right] \to \frac{1}{R}\left(\frac{1}{s^2} + \frac{1}{s(1-e^{-s})}\right)\frac{e^{st}}{1 + sL/R}$$

NEAR $s = 0$ $$\frac{1}{R}\left(\frac{1}{s^2} + \frac{1}{s\left(-\frac{s^2}{2} - s\right)}\right)\frac{e^{st}}{1 + sL/R} \to \frac{1}{Rs^2}\left(1 - \frac{1}{1 + s/2}\right)(1 + st + \ldots)(1 - sL/R + \ldots)$$

$$\to \frac{1}{Rs^2}\left(1 - (1 - s/2)\right)(1 + st + \ldots)(1 - sL/R)$$

$$\to \frac{1}{Rs^2}\left(\frac{s}{2}\right)(1 + \ldots)(1 \ldots) = \frac{1}{2Rs} + \ldots$$

THEREFORE $s = 0$ IS ACTUALLY A POLE OF ORDER ONE AND

$$\mathrm{RES}\left[J(s) e^{st} ; 0\right] = \frac{1}{2R}.$$

(iii) NOW CALCULATE THE RESIDUE AT $s = 2\pi i k$ $k = \pm 1, \pm 2, \ldots$

WE USE P/Q METHOD

$$J(s)e^{st} = \left(\frac{(1-e^s)+s}{s^2(1-e^s)}\right)\frac{e^{st}}{sL+R} = \frac{P}{Q} \qquad Q = 1-e^s.$$

NOW
$$\text{RES}\left[Je^{st}; 2\pi i k\right] = \left.\frac{s\,e^{st}}{s^2(sL+R)(-e^s)}\right|_{2\pi i k} = \frac{-e^{2\pi i k t}}{(2\pi i k)^2 L + 2\pi i k R}$$

THIS GIVES IN TOTAL FROM (*) THAT

$$I(t) = e^{-Rt/L}\left(\frac{L}{R^2} - \frac{1}{R(1-e^{-R/L})}\right) + \frac{1}{2R} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\left(\frac{-e^{2\pi i k t}}{(2\pi i k)^2 L + 2\pi i k R}\right)$$

THIS CAN BE WRITTEN AS

$$I(t) = e^{-Rt/L}\left(\frac{L}{R^2} - \frac{1}{R(1-e^{-R/L})}\right) + \frac{1}{2R} - \sum_{k=1}^{\infty}\left(\frac{e^{2\pi i k t}}{-4\pi^2 k^2 L + 2\pi i k R} + \frac{e^{-2\pi i k t}}{-4\pi k^2 L - 2\pi i k R}\right)$$

LET $Z_k = \dfrac{e^{2\pi i k t}}{-4\pi k^2 L + 2\pi i k R}$ THEN $\displaystyle\sum_{k=1}^{\infty}(Z_k + \bar{Z}_k) = 2\sum_{k=1}^{\infty}\text{RE}(Z_k)$

SO
$$I(t) = e^{-Rt/L}\left(\frac{L}{R^2} - \frac{1}{R(1-e^{-R/L})}\right) + \frac{1}{2R} + 2\sum_{k=1}^{\infty}\text{RE}\left[\frac{e^{2\pi i k t}}{-4\pi^2 k^2 L + 2\pi i k R}\right]$$

NOW
$$\text{RE}\left[\frac{e^{2\pi i k t}}{-4\pi^2 k^2 L + 2\pi i k R}\right] = \text{RE}\left[\frac{(\cos(2\pi k t) + i\sin(2\pi k t))(-4\pi^2 k^2 L - 2\pi i k R)}{16\pi^4 k^4 L^2 + 4\pi^2 k^2 R^2}\right]$$

THIS YIELDS
$$I(t) = e^{-Rt/L}\left(\frac{L}{R^2} - \frac{1}{R(1-e^{-R/L})}\right) + \frac{1}{2R} - 2\sum_{k=1}^{\infty}\left(\frac{-4\pi^2 k^2 L\cos(2\pi k t) + 2\pi k R\sin(2\pi k t)}{16\pi^4 k^4 L^2 + 4\pi^2 k^2 R^2}\right)$$

NOTICE THAT THE SAWTOOTH FUNCTION GIVES RISE TO INFINITE # OF MODES OF OSCILLATION.