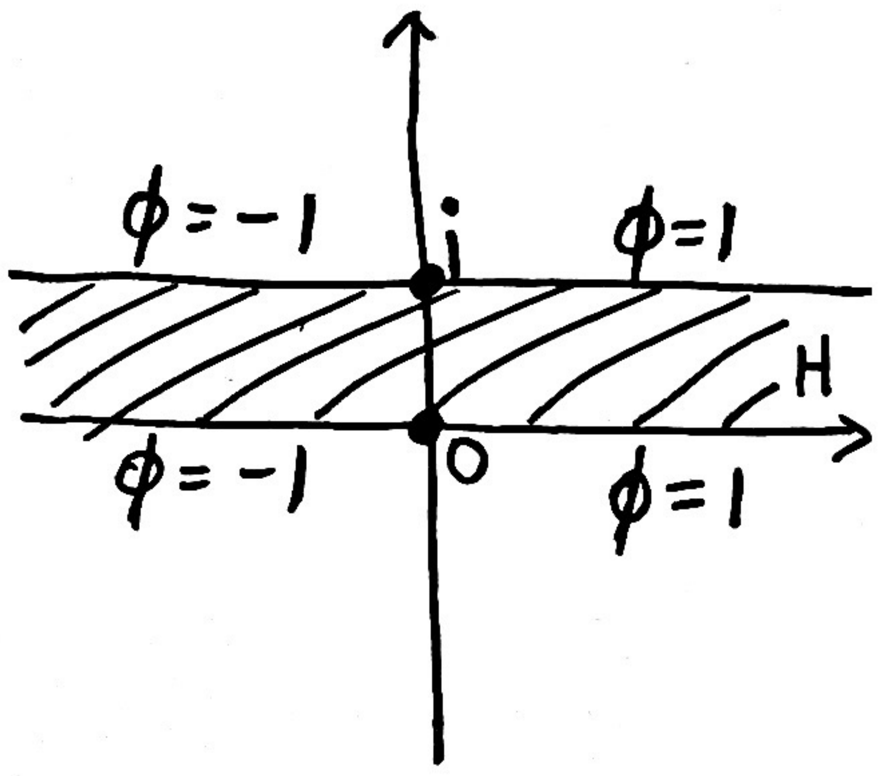
5 pts 1. Let H = {z : 0 < Im z < 1 }. Find a function ϕ that is harmonic in H and satisfies ϕ = -1 on {x $\leqslant$ 0} and { x+i : x $\leqslant$ 0} as well as ϕ = 1 on {x > 0} and { x+i : x > 0}. Hint: Consider the mapping f(z) = $(e^{\pi})^z$.



Map the domain H using $f(z) = e^{\pi z}$:

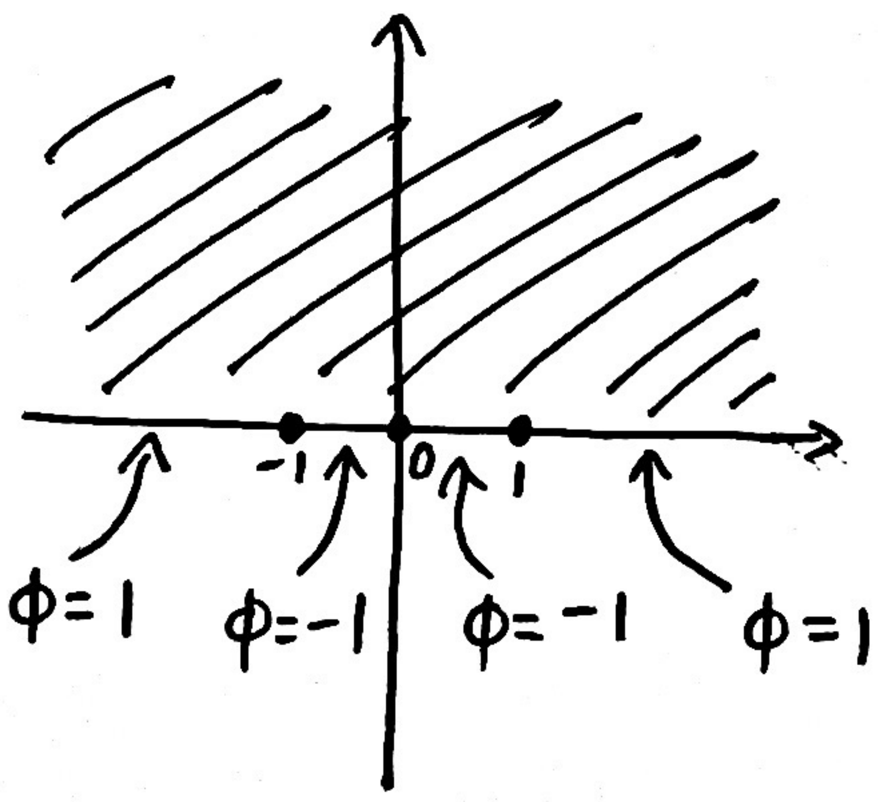
$$\left.\begin{array}{l} \text{Re } z \leq 0, \text{ Im } z = 0 \rightarrow 0 < \text{Re } w \leq 1, \text{ Im } w = 0 \\ \text{Re } z \leq 0, \text{ Im } z = 1 \rightarrow -1 \leq \text{Re } w < 0, \text{ Im } w = 0 \end{array}\right\} \phi = -1$$

$$\left.\begin{array}{l} \text{Re } z > 0, \text{ Im } z = 0 \rightarrow \text{Re } w > 1, \text{ Im } w = 0 \\ \text{Re } z > 0, \text{ Im } z = 1 \rightarrow \text{Re } w < -1, \text{ Im } w = 0 \end{array}\right\} \phi = 1$$

The boundary of H maps to the real line

For an interior point like $z = \frac{i}{2}$, $w = e^{i\pi/2} = i$ is in the upper half plane

Thus H maps to the upper half plane:



On the real axis, there are discontinuity points at $x = 1$ and $x = -1$

ϕ is in the form:

$$\phi(w) = a + b\,\text{Arg}(w-1) + c\,\text{Arg}(w+1)$$

$w \in (-\infty, -1)$: $\phi(w) = a + b\pi + c\pi = 1$

$w \in (-1, 1)$: $\phi(w) = a + b\pi = -1$

$w \in (1, \infty)$: $\phi(w) = a = 1$

$\Rightarrow c = \frac{2}{\pi}$, $b = -\frac{2}{\pi}$, $a = 1$

$$\phi(w) = 1 - \frac{2}{\pi}\text{Arg}(w-1) + \frac{2}{\pi}\text{Arg}(w+1)$$

To map back to the original picture, a harmonic solution is

$$\boxed{\phi(z) = 1 - \frac{2}{\pi}\text{Arg}(e^{\pi z} - 1) + \frac{2}{\pi}\text{Arg}(e^{\pi z} + 1)}$$

5 pts

2. Use the Fourier transform to solve the equation

$u_t = Du_{xx} \quad -\infty < x < \infty \quad t > 0 \quad$ with $u(x, 0) = f(x)$

$$U(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx \quad \text{and} \quad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,t)e^{ikx}\,dk.$$

By the Fourier Transform,

$$U_t = D(ki)^2 U = -Dk^2 U$$

since $\mathcal{F}[u_x(x,t)] = \int_{-\infty}^{\infty} u_x(x,t)e^{-ikx}\,dx = \underbrace{u(x,t)e^{-ikx}\Big|_{-\infty}^{\infty}}_{=0 \text{ assuming } u \to 0 \text{ as } |x| \to \infty} + ik\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx = ik\,\mathcal{F}[u(x,t)]$

Then $U = \hat{f}(k)e^{-Dk^2t}$

where $\hat{f}(k) = \mathcal{F}[f(x)]$

Inverse Fourier Transform:

By convolution theorem,

$$u = f * \mathcal{F}^{-1}[e^{-Dk^2t}]$$

$$\mathcal{F}^{-1}[e^{-Dk^2t}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-Dk^2t+ikx}\,dk$$

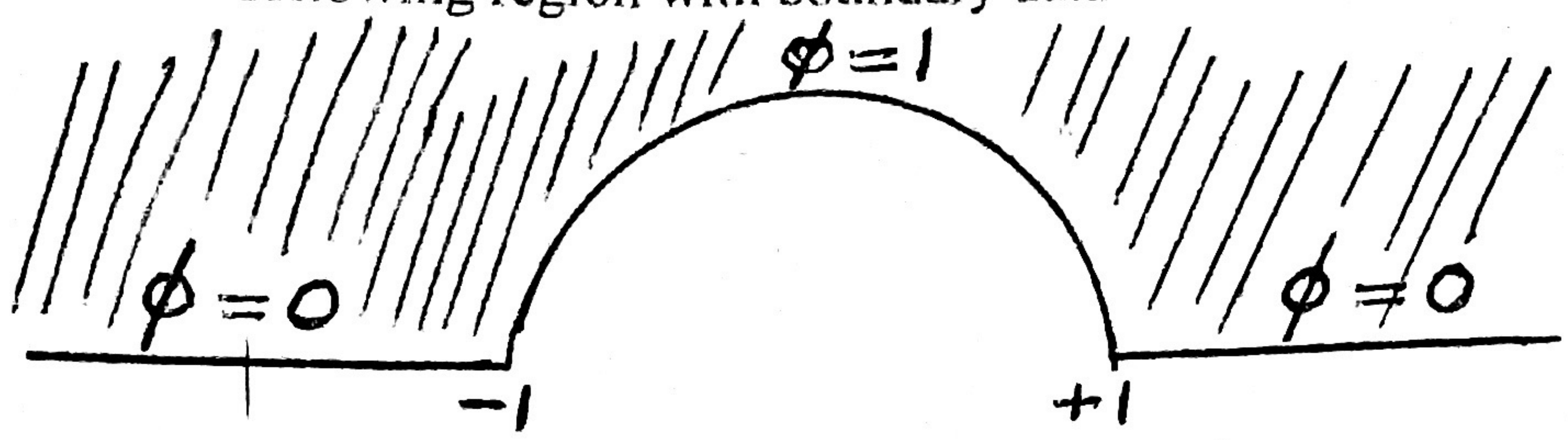
$$= \frac{1}{2\pi}e^{-\frac{x^2}{4tD}}\int_{-\infty}^{\infty} e^{\underbrace{-Dk^2t+ikx+\frac{x^2}{4tD}}_{-\frac{t}{D}\left(Dk-\frac{ix}{2t}\right)^2}}\,dk$$

$$= \frac{1}{2\pi}e^{-\frac{x^2}{4tD}}\int_{-\infty}^{\infty} e^{-Dk^2t}\,dk \quad \text{(shifting property)}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{Dt}}\,e^{-\frac{x^2}{4tD}}$$

Then $\boxed{u(x) = \frac{1}{2\sqrt{\pi Dt}}\int_{-\infty}^{\infty} f(x')\,e^{-\frac{(x-x')^2}{4tD}}\,dx'}$

3. Consider the following region with boundary data.



(a) Show that the Joukowski map J(z) = (1/2)(z + 1/z) maps the shaded region onto the upper half plane.

(b) Solve Laplaces equation in the shaded region, with the indicated boundary conditions.

$\boxed{2\text{pts}}$ (a) The line $[1,\infty)$ maps onto $[1,\infty)$

The line $[-\infty,-1]$ maps onto $(-\infty,-1]$

The semicircle $e^{i\theta}$ $(0\le\theta\le\pi)$ maps to

$$\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)=\cos\theta$$

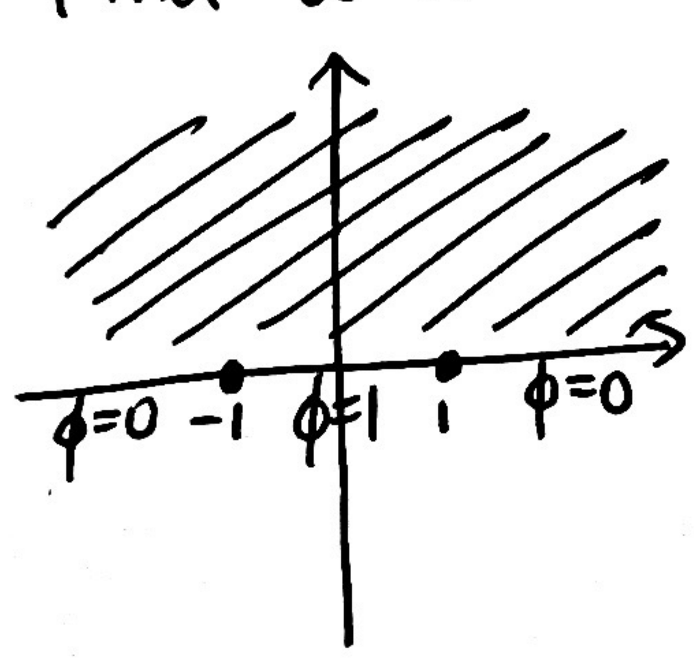
$\Rightarrow$ onto $[-1,1]$

For an interior point like $z=2i$,

$$J(z)=\frac{1}{2}\left(2i+\frac{1}{2i}\right)=\frac{1}{2}\left(2i-\frac{1}{2}i\right)=\frac{3}{4}i$$ is on the upper half plane

$\therefore$ Boundary maps to real line and interior maps to the upper half plane.

$\boxed{3\text{pts}}$ (b) Find a harmonic function with the conditions:



$$\phi(w)=a+b\,\mathrm{Arg}(w-1)+c\,\mathrm{Arg}(w+1)$$

For $w<-1$	$a+b\pi+c\pi$	$=0$
$-1<w<1$	$a+b\pi$	$=1$
$w>1$	a	$=0$

$\Rightarrow b=\frac{1}{\pi},\ c=-\frac{1}{\pi}$

$$\Rightarrow \phi(w)=\frac{1}{\pi}\mathrm{Arg}(w-1)-\frac{1}{\pi}\mathrm{Arg}(w+1)$$

On the original picture,

$$\phi(z)=\frac{1}{\pi}\mathrm{Arg}\left(\frac{1}{2}\left(z+\frac{1}{z}\right)-1\right)-\frac{1}{\pi}\mathrm{Arg}\left(\frac{1}{2}\left(z+\frac{1}{z}\right)+1\right)$$

4. Let $w = f(z)$ be the Mobius transformation mapping the points $0, \lambda, \infty$ to $-i, 1, i$, respectively, where λ is real. For what values of λ is the upper half plane mapped onto $|w| < 1$?

$\boxed{\text{5 pts}}$

From the points of the Möbius transform, the real line (containing $0, \lambda, \infty$) maps to the unit circle (containing $-i, 1, i$).

A Möbius transformation is of the form

$$z \mapsto \frac{iz - ai}{z+a} \quad \text{with} \quad \frac{i\lambda - ai}{\lambda + a} = 1$$

$$\Rightarrow i\lambda - ai = \lambda + a$$

$$(i-1)\lambda = (1+i)a$$

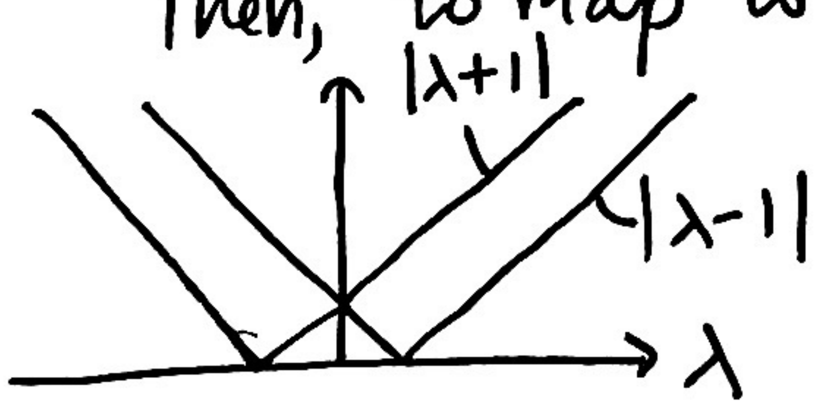
$$a = \lambda\left(\frac{-1+i}{1+i}\right) = \lambda i$$

$$\Rightarrow z \mapsto \frac{iz+\lambda}{z+\lambda i}$$

Take a point in the upper half plane such as $z = i$, then

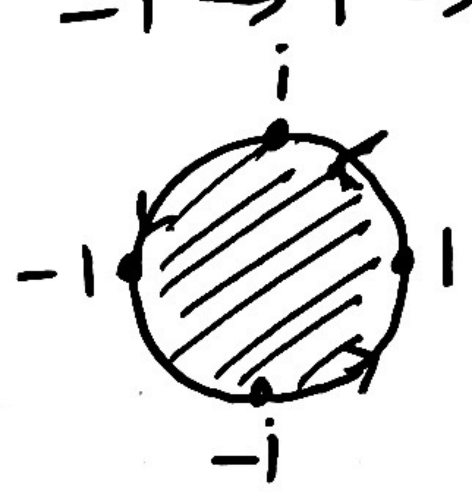
$$i \mapsto \frac{-1+\lambda}{i+\lambda i} \quad \text{and magnitude is} \quad \frac{|\lambda - 1|}{|\lambda+1|}$$

Then, to map to $|w|<1$, we need $|\lambda-1| < |\lambda+1|$



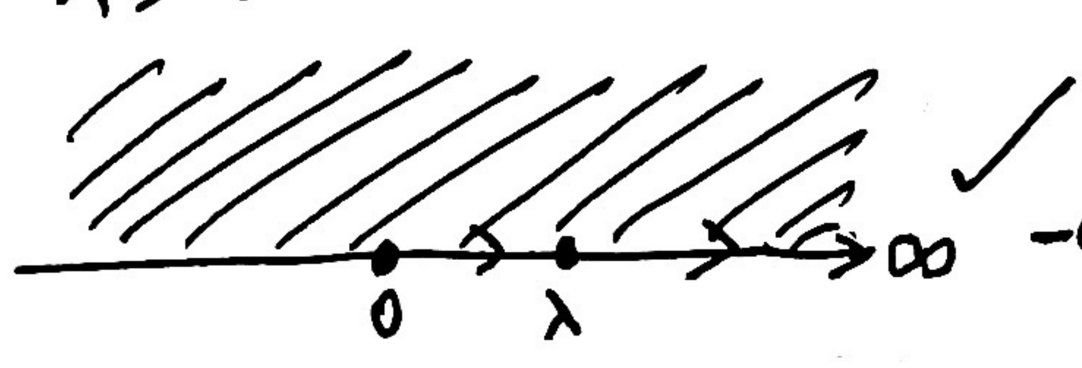
$\Rightarrow \boxed{\lambda > 0}$.

Alternatively, traversing the unit circle in the order $-i \to 1 \to i \to -i$ puts the interior on the left hand side of the traversed path.

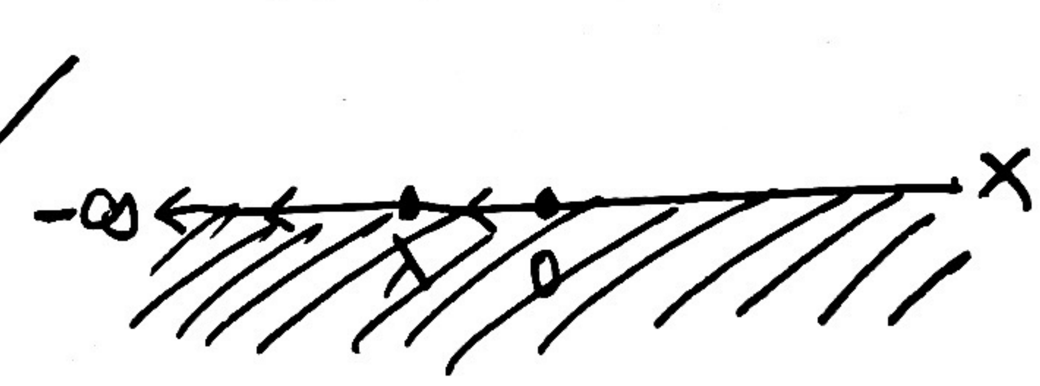


We want the interior on the left when traversing $0 \to \lambda \to \infty$ along the real line, interior on upper half plane. Then:

$\lambda > 0$:



$\lambda < 0$:



$\Rightarrow \boxed{\lambda > 0}$.