1. Find the Laurent series for

2pts (a) $(\sin 2z)/(z^3)$ in $|z|>0$

Recall: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$

$\therefore \sin 2z = 2z - \frac{2^3}{3!}z^3 + \frac{2^5}{5!}z^5 - \frac{2^7}{7!}z^7 + \cdots$

$$\frac{\sin 2z}{z^3} = \frac{2}{z^2} - \frac{2^3}{3!} + \frac{2^5}{5!}z^2 - \frac{2^7}{7!}z^4 + \cdots$$

$$\boxed{= \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} z^{2n-2} \cdot (-1)^n}$$

3pts (b) $1/(z+z^2)$ for $1<|z|$ → Centered at $z=0$.

$$\frac{1}{z+z^2} = \frac{1}{z} \bullet \frac{1}{1+z}$$

For $|z|>1$, $\frac{1}{1+z} = \frac{1/z}{1+1/z} \overset{\text{Geometric Series}}{=} \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right)$

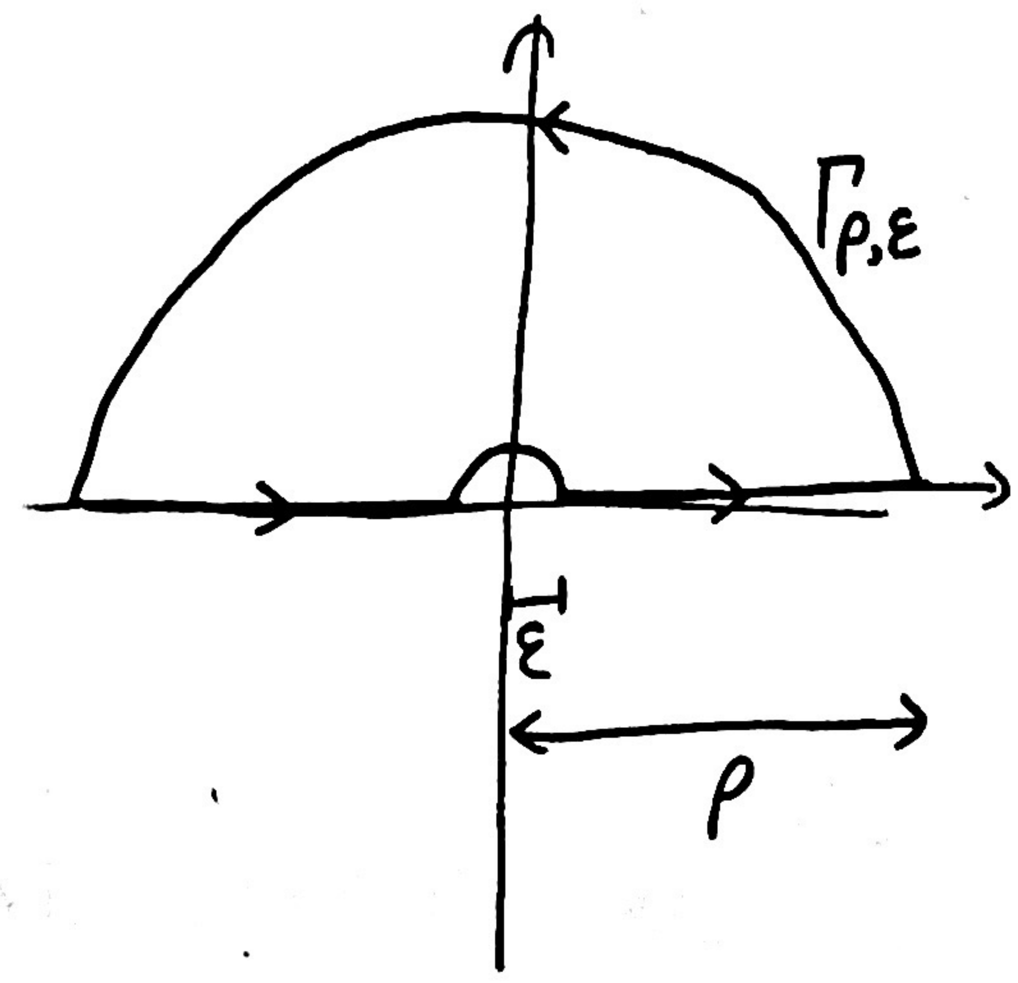
$$\frac{1}{z+z^2} = \frac{1}{z^2}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+2}}}$$

converges since $\left|\frac{1}{z}\right| < 1$

$\boxed{5 \text{ pts}}$ 2. Evaluate

$$\int_0^\infty \frac{(\cos x) - 1}{x^2}\, dx$$

Take the integral of $f(z) = \dfrac{e^{iz}-1}{z^2}$ along the contour



$\oint_{\Gamma_{\rho,\varepsilon}} f(z)\,dz = 0$ since there are no singularities in the contour

Integral decomposes into

$$\underset{①}{\int_{-\rho}^{-\varepsilon}} + \underset{②}{\int_{\varepsilon}^{\rho}} + \underset{③}{\int_{\text{small arc}}} + \underset{④}{\int_{\text{large arc}}}$$

Taking $\varepsilon \to 0$, $\rho \to 0$,

$① \to \int_{-\infty}^{0} \frac{e^{ix}-1}{x^2}\,dx = \int_0^\infty \frac{e^{-ix}-1}{x^2}\,dx$

$② \to \int_0^\infty \frac{e^{ix}-1}{x^2}\,dx$

NOTE: $\dfrac{e^{iz}-1}{z^2} = \dfrac{iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \cdots}{z^2}$ So the pole is order 1 not 2

$③ \to -i\pi \operatorname{Res}(f;0) = -i\pi \left.\frac{d}{dz}\right|_{z=0} (e^{iz}-1) = -i\pi \cdot i = \pi$

$④ \to 0$ by Jordan's Lemma $\Rightarrow \int_0^\infty \frac{2\cos x - 2}{x^2} + \pi = 0$

$$\Rightarrow \int_0^\infty \frac{\cos x - 1}{x^2} = -\frac{\pi}{2}$$

3. Compute the following

$\boxed{1pt}$ (a) All values of $(-i)^{1+i}$

$$(-i)^{1+i} = e^{(2k\pi-\pi/2)i(1+i)} = e^{(2k\pi-\pi/2)(-1+i)}$$
$$= -i\cdot e^{\pi/2+2k\pi} \quad \text{for all } k\in\mathbb{Z}$$

$\boxed{1pt}$ (b) All values of $2^{\pi i}$

$$2^{\pi i} = e^{(\mathrm{Log}\,2+2k\pi i)\pi i} = e^{-2k\pi^2}e^{(\mathrm{Log}\,2)\pi i}$$
$$= e^{-2k\pi^2}\left(\cos(\pi\,\mathrm{Log}\,2)+i\sin(\pi\,\mathrm{Log}\,2)\right)$$
for all $k\in\mathbb{Z}$

$\boxed{3pts}$ (c) Find all solutions z of $\cos z = ki$ for k a positive real number.

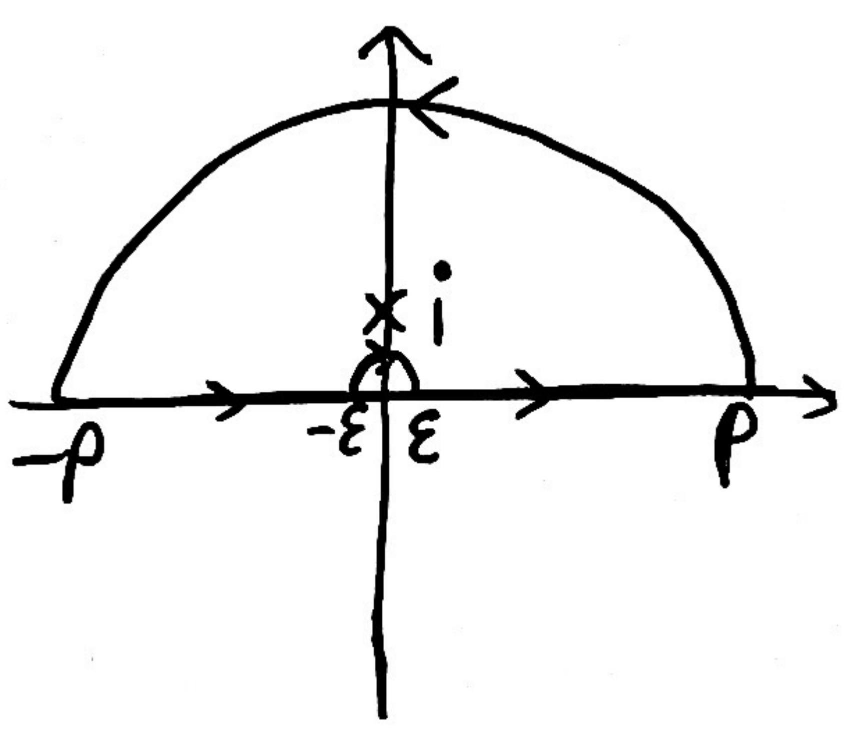
$$\frac{e^{iz}+e^{-iz}}{2} = ki$$
$$e^{2iz}+1-2ki\,e^{iz} = 0$$
$$e^{iz} = \frac{2ki+(-4k^2-4)^{1/2}}{2} = i\left(k+(1+k^2)^{1/2}\right)$$
$$z = -i\left(\mathrm{Log}\left(k+(1+k^2)^{1/2}\right)+\frac{\pi}{2}i+2\pi\ell i\right) \quad \text{for all } \ell\in\mathbb{Z}$$
$$= \left(\frac{\pi}{2}+2\pi\ell\right)-i\,\mathrm{Log}\left(k+(1+k^2)^{1/2}\right)$$

where $z^{1/2}$ refers to both values, $+z^{1/2}$ and $-z^{1/2}$

$\boxed{5\text{pts}}$ 4. Evaluate

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx$$

Integrate the function $\dfrac{\operatorname{Log} z}{z^2+1}$ on the contour



$$\oint_{\Gamma_\rho} \frac{\operatorname{Log} z}{z^2+1} = \int_\varepsilon^\rho + \int_{-\rho}^{-\varepsilon} + \int_{\substack{\text{large}\\\text{arc}}} + \int_{\substack{\text{small}\\\text{arc}}}$$

As $\rho \to \infty$, $\varepsilon \to 0$,

$\int_{\substack{\text{large}\\\text{arc}}} \to 0$ since $\dfrac{\operatorname{Log} \rho}{\rho^2-1} \cdot \rho \to 0$

$$\int_\varepsilon^\rho \to \int_0^\infty \frac{\operatorname{Log} x}{x^2+1}\,dx$$

$\int_{\substack{\text{small}\\\text{arc}}} \to 0$ since $\dfrac{\operatorname{Log} \varepsilon}{1-\varepsilon^2} \cdot \varepsilon \to 0$

$(1-\varepsilon^2 \not\to 0\ (\to 1)$, $\varepsilon \operatorname{Log} \varepsilon \to 0$,

L'Hopital's Rule: $\lim\limits_{\varepsilon\to0} \varepsilon \operatorname{Log} \varepsilon = \lim\limits_{\varepsilon\to0} \frac{1/\varepsilon}{-1/\varepsilon^2} = 0)$

$$\int_{-\rho}^{-\varepsilon} \to \int_{-\infty}^0 \frac{\operatorname{Log} x}{x^2+1}\,dx = \int_{-\infty}^0 \frac{\operatorname{Log}|x| + \pi i}{x^2+1}\,dx = \int_0^\infty \frac{\operatorname{Log} x + \pi i}{x^2+1}\,dx$$

By Residue Theorem,

$$\operatorname{Res}(f; i) = \frac{\operatorname{Log} i}{i+i} = \frac{\frac{\pi}{2} i}{2i} = \frac{\pi}{4}$$

$$\Rightarrow \text{Integral} = \frac{\pi^2}{2} i = 2\int_0^\infty \frac{\operatorname{Log} x}{x^2+1}\,dx + \int_0^\infty \frac{\pi i}{x^2+1}\,dx$$

Equating real parts, $\displaystyle\int_0^\infty \frac{\operatorname{Log} x}{x^2+1}\,dx = 0$