MATH 301 HOMEWORK 9

PROBLEM 1 CONSIDER $y'' + w_0^3 y = f(t)$ $\quad w_0 > 0$

$$y(0) = y'(0) = y''(0) = 0.$$

CALCULATE THE LAPLACE TRANSFORM SOLUTION IN THE FORM $y(t) = \int_0^t h(\tau) f(t-\tau)\, d\tau$ FOR SOME $h(t)$ TO BE FOUND.

PROBLEM 2 FIND THE INVERSE TRANSFORMS OF

(i) $F(s) = e^{-a\sqrt{s}}$ FOR $a > 0$

(ii) $F(s) = \dfrac{1}{\sqrt{s}} e^{-a\sqrt{s}}$ FOR $a > 0$

(iii) $F(s) = \dfrac{1}{\sqrt{s^2 + w^2}}$ $\quad$ (show that $f(t) = \dfrac{2}{\pi} \displaystyle\int_0^1 \dfrac{\cos(t w p)}{\sqrt{1-p^2}}\, dp$)

PROBLEM 3 SUPPOSE $f(t)$ IS PERIODIC WITH PERIOD $2T$ AND IS ODD WITH PERIOD T. IN OTHER WORDS $f(t+2T) = f(t)$, $f(t+T) = -f(t)$.

(i) SHOW THAT $F(s) = \mathcal{L}(f(t)) = \dfrac{F_0(s)}{1 + e^{-sT}}$ $\quad F_0(s) = \displaystyle\int_0^T f(t) e^{-st}\, dt$.

(ii) FIND A FUNCTION $f(t)$ WHOSE TRANSFORM IS $F(s) = \dfrac{1}{s} \tanh\left(sT/2\right)$.

PROBLEM 4 SOLVE THE FOLLOWING WAVE EQUATION USING FOURIER TRANSFORMS:

$$u_{tt} + 2u_t + u = u_{xx} \qquad t \geq 0 \quad -\infty < x < \infty$$

$$u_t(x,0) = 0 \qquad u(x,0) = \begin{cases} e^{-x} & x > 0 \\ e^{x} & x < 0 \end{cases}$$

$$u, u_x \to 0 \text{ AS } |x| \to \infty.$$

PROBLEM 5 FIND THE NUMBER OF ZEROES OF

(i) $p(z) = z^3 + 2z^2 + z + 1$ IN Right-Half PLANE

(ii) $p(z) = z^4 + 2z^3 + 3z^2 + z + 2$ IN Right-Half plane.

PROBLEM 6 SOLVE THE HEAT-CONDUCTION PROBLEM

$$u_t = D u_{xx} \qquad 0 \le x < \infty,\ t > 0$$

$$u(x,0) = 0, \qquad u(0,t) = h(t)$$ USING LAPLACE TRANSFORMS.

PROBLEM 7 GIVEN THE WAVE EQUATION

$$(*)\quad \begin{cases} u_{tt} - u_{xx} = 0 \\ u(x,0) = 0, \quad u_t(x,0) = 0 \\ u(0,t) = 0 \quad u(L,t) = f(t) \end{cases}$$

(i) SHOW THAT THE LAPLACE TRANSFORM OF THE SOLUTION IS

$$U(x,s) = \hat{F}(s)\,\frac{\sinh(sx)}{\sinh(sL)} \qquad \text{WHERE} \qquad \hat{F}(s) = \mathcal{L}(f(t))$$

(ii) LET $\phi_s(x,t)$ BE THE SOLUTION TO $(*)$ WHEN $f(t) = 1$.

SHOW THAT THE GENERAL SOLUTION IS (WHEN $f(t) \neq 1$)

$$u(x,t) = \int_0^t \frac{\partial \phi_s}{\partial \tau}(x,\tau)\, f(t-\tau)\, d\tau.$$

PROBLEM 8 CONSIDER THE FOLLOWING INITIAL VALUE PROBLEM FOR $y(t)$

$$y'''' + K y''' + y'' + y' = e^{-t} \qquad t \ge 0, \qquad K \ge 0 \text{ IS REAL}$$

$$y(0) = 0,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 4.$$

(i) CALCULATE THE LAPLACE TRANSFORM OF $y(t)$, DENOTED BY $Y(s)$, IN THE FORM $Y(s) = \dfrac{P(s)}{Q(s)}$ WHERE P, Q polynomials

(ii) IF $K = 2$, prove that $y(t)$ is bounded as $t \to \infty$ AND calcul[ate] $\lim_{t \to \infty} y(t)$.

(iii) FOR what range of K is $y(t)$ bounded as $t \to \infty$.

(iv) FIND the behavior of $y(t)$ as $t \to \infty$ when $K = 1$. what happe[ns] FOR $0 < K < 1$?