Homework 9 Solutions

Q1 4 pts

$$\mathcal{L}\{y''\} = \int_0^\infty y''(t)\, e^{-st}\, dt = y'(t)e^{-st}\Big|_0^\infty + \int_0^\infty s\, y'(t) e^{-st}\, dt$$

$$= -y'(0) + s y(t) e^{-st}\Big|_0^\infty + \int_0^\infty s^2 y(t)\, e^{-st}\, dt$$

$$= -y'(0) - s y(0) + s^2 Y(s)$$

Laplace transform the ODE:

$$s^2 Y(s) + \omega_0^3\, Y(s) = F(s) \left(:= \mathcal{L}\{f\} \right)$$

$$Y(s) = \frac{F(s)}{s^2 + \omega_0^3} = \int_0^\infty y(t)\, e^{-st}\, dt$$

By convolution property, $y(t) = f(t) * \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^3}\right\}$

$$= \int_0^t h(t)\, f(\tau - t)\, d\tau$$

where $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^3}\right\} = \frac{1}{\omega_0^{3/2}} \sin\left(\omega_0^{3/2}\, t\right)$

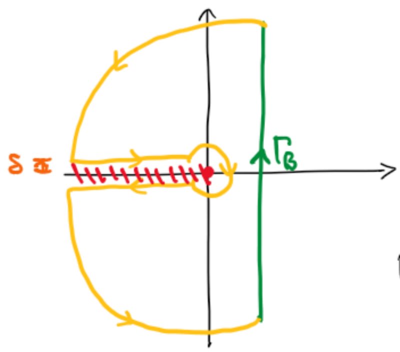
[Laplace Transform of $\sin(\omega t) = \frac{\omega}{s^2+\omega^2}$]

Q2

(i) According to the inversion formula,

3 pts $$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-a\sqrt{s}} e^{st}\, ds \quad \text{with } \gamma>0$$

Given $\sqrt{s}$ is a multi-valued function, use the principal branch cut for $\sqrt{s}$ and integrate $e^{-a\sqrt{s}}e^{st}$ over the contour:



This function is analytic inside this contour,

so $$f(t) = \frac{1}{2\pi i}\int_{\Gamma_B} = -\frac{1}{2\pi i}\left(\int_{\text{large arc}} + \int_{\text{small arc}} + \int_{-\infty+\delta i}^{\delta i} - \int_{-\infty-\delta i}^{-\delta i}\right)$$

We know: $\int_{\text{small arc}} \to 0$ since $e^{-a\sqrt{s}}e^{st} \to 1$, length $\to 0$

$\int_{\text{large arc}} \to 0$

Then we are left with $-\frac{1}{2\pi i}\left(\int_{-\infty+\delta i}^{\delta i} - \int_{-\infty-\delta i}^{-\delta i}\right)$

By taking $\delta \to 0$, this becomes

$$-\frac{1}{2\pi i}\left(\int_0^\infty e^{-ai\sqrt{s}}e^{-st}\,ds - \int_0^\infty e^{ai\sqrt{s}}e^{-st}\,ds\right)$$

$$= -\frac{1}{2\pi i}\left(\int_0^\infty e^{-air-r^2t}\,2r\,dr - \int_0^\infty e^{air-r^2t}\,2r\,dr\right) \quad (\text{substitute } s=r^2)$$

$$= -\frac{1}{\pi i}\left(\int_0^\infty r e^{-t(r+\frac{ai}{2t})^2-\frac{a^2}{4t}}\,dr - \int_0^\infty r e^{-t(r-\frac{ai}{2t})^2-\frac{a^2}{4t}}\,dr\right)$$

$$= -\frac{1}{\pi i}e^{-\frac{a^2}{4t}}\left(\int_0^\infty \left(r+\frac{ai}{2t}\right)e^{-t(r+\frac{ai}{2t})^2}\,dr - \int_0^\infty \frac{ai}{2t}e^{-t(r+\frac{ai}{2t})^2}\,dr - \int_0^\infty\left(r-\frac{ai}{2t}\right)e^{-t(r-\frac{ai}{2t})^2}\,dr - \int_0^\infty \frac{ai}{2t}e^{-t(r-\frac{ai}{2t})^2}\,dr\right)$$

$$= -\frac{1}{\pi i}e^{-\frac{a^2}{4t}}\left(-\frac{1}{2t}e^{-t(r+\frac{ai}{2t})^2}\Big|_0^\infty + \frac{1}{2t}e^{-t(r-\frac{ai}{2t})^2}\Big|_0^\infty - \int_0^\infty \frac{ai}{t}e^{-tr^2}\,dr\right) \quad (\text{shifting property of Gaussian integral})$$

$$= \frac{1}{\pi i}e^{-\frac{a^2}{4t}}\frac{ai}{2t}\sqrt{\frac{\pi}{t}} = \frac{a}{2\sqrt{\pi t^3}}e^{-\frac{a^2}{4t}}$$

(ii) Using the same contour and branch cut as before, integral becomes

3 pts

$$-\frac{1}{2\pi i}\left(\int_{-\infty+\delta i}^{\delta i}\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}e^{st}\,ds-\int_{-\infty-\delta i}^{-\delta i}\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}e^{st}\,ds\right)$$

$$\xrightarrow{\delta\to 0}-\frac{1}{2\pi i}\left(\int_0^\infty\frac{1}{i\sqrt{s}}e^{-ai\sqrt{s}}e^{-st}\,ds-\int_0^\infty\frac{1}{-i\sqrt{s}}e^{ai\sqrt{s}}e^{-st}\,ds\right)$$

$$=-\frac{1}{2\pi i}\left(\int_0^\infty\frac{1}{ir}e^{-air}e^{-r^2t}\,2r\,dr-\int_0^\infty\frac{1}{-ir}e^{air}e^{-r^2t}\,2r\,dr\right)\quad(s=r^2)$$

$$=\frac{1}{\pi}\int_0^\infty e^{-air-r^2t}\,dr-\frac{1}{\pi}\int_0^\infty e^{air-r^2t}\,dr$$

$$=\frac{1}{\pi}\int_0^\infty e^{-t\left(r+\frac{ai}{2t}\right)^2-\frac{a^2}{4t}}\,dr-\frac{1}{\pi}\int_0^\infty e^{-t\left(r-\frac{ai}{2t}\right)^2-\frac{a^2}{4t}}\,dr$$

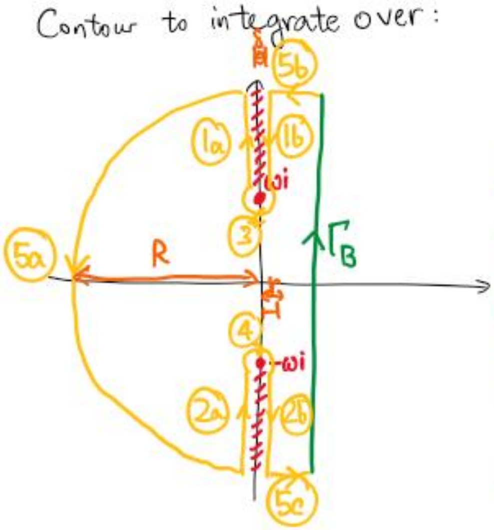
$$=\frac{2}{\pi}e^{-\frac{a^2}{4t}}\int_0^\infty e^{-tr^2}\,dr=\frac{1}{\pi}\sqrt{\frac{\pi}{t}}\,e^{-\frac{a^2}{4t}}=\frac{1}{\sqrt{\pi t}}\,e^{-\frac{a^2}{4t}}$$

(iii) Use the inversion formula with Bromwich contour $\gamma > 0$

3pts Branch cuts of $\frac{1}{\sqrt{s^2+\omega^2}}$ lie where $s^2+\omega^2$ is real negative:

$\omega = ai$ where $a \in \mathbb{R}$, $|a| > \omega$

Contour to integrate over:



Integrate $\int_\Gamma \frac{1}{\sqrt{s^2+\omega^2}} e^{st}\, ds = 0$ since analytic

By taking $\delta \to 0$, $R \to \infty$, $r \to 0$:

$$1a \longrightarrow i\int_\omega^\infty \frac{1}{-i\sqrt{s^2-\omega^2}} e^{ist}\, ds$$

$$1b \longrightarrow -i\int_\omega^\infty \frac{1}{i\sqrt{s^2-\omega^2}} e^{ist}\, ds$$

$$2a \longrightarrow i\int_\omega^\infty \frac{1}{i\sqrt{s^2-\omega^2}} e^{-ist}\, ds$$

$$2b \longrightarrow -i\int_\omega^\infty \frac{1}{-i\sqrt{s^2-\omega^2}} e^{-ist}\, ds$$

$$3 \longrightarrow -\int_{-3\pi/2}^{\pi/2} (rie^{i\theta}) \frac{1}{\sqrt{(\omega i + re^{i\theta})^2+\omega^2}} e^{(\omega i + re^{i\theta})t}\, d\theta$$

$$4 \longrightarrow -\int_{-\pi/2}^{3\pi/2} (rie^{i\theta}) \frac{1}{\sqrt{(-\omega i + re^{i\theta})^2+\omega^2}} e^{(-\omega i + re^{i\theta})t}\, d\theta$$

$$\Rightarrow 3 + 4 \longrightarrow 0$$

$$5a, 5b, 5c \longrightarrow 0$$

$$\Rightarrow f(t) = \frac{1}{2\pi i}\int_{\Gamma_B} = -\frac{1}{2\pi i}\left(1a + 1b + 2a + 2b\right) = \frac{1}{\pi i}\int_\omega^\infty \frac{1}{\sqrt{s^2-\omega^2}} e^{ist}\, ds - \frac{1}{\pi i}\int_\omega^\infty \frac{1}{\sqrt{s^2-\omega^2}} e^{-ist}\, ds$$

$$= \int_\omega^\infty \frac{2\sin(st)}{\pi\sqrt{s^2-\omega^2}}\, ds$$

Substitute $s = \omega\rho$

$$= \int_1^\infty \frac{2\sin(\omega\rho t)}{\pi\sqrt{\omega^2(\rho^2-1)}}\,\omega\, d\rho = \int_1^\infty \frac{2\sin(\omega\rho t)}{\pi\sqrt{\rho^2-1}}\, d\rho = \int_0^1 \frac{2\cos(\omega\rho t)}{\pi\sqrt{1-\rho^2}}\, d\rho \quad \square$$

Q3 (i) Split the Laplace integral into blocks of length T:

2 pts

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \sum_{k=0}^{\infty}\int_{kT}^{(k+1)T} f(t)e^{-st}\,dt$$

Applying the property,

$$\int_{kT}^{(k+1)T} f(t)e^{-st}\,dt = \int_0^T f(t+kT)\,e^{-s(t+kT)}\,dt$$

$$= \begin{cases} \int_0^T f(t)e^{-s(t+kT)}\,dt & \text{if } k \text{ even} \\ -\int_0^T f(t)\,e^{-s(t+kT)}\,dt & \text{if } k \text{ odd} \end{cases}$$

$$\Rightarrow F(s) = \sum_{k=0}^{\infty}(-1)^k \int_0^T f(t)e^{-s(t+kT)}\,dt$$

$$= \int_0^T f(t)\frac{e^{-st}}{1+e^{-sT}}\,dt \quad \left(\text{geometric series } \sum_{k=0}^{\infty}(-1)^k e^{-s(t+kT)} = \frac{e^{-st}}{1+e^{-sT}}\right)$$

$$= \frac{F_0(s)}{1+e^{-sT}} \quad \text{where } F_0(s) = \int_0^T f(t)e^{-st}\,dt$$

(ii) $F(s) = \frac{1}{s}\tanh\left(\frac{sT}{2}\right) = \frac{1}{s}\frac{e^{sT/2}-e^{-sT/2}}{e^{sT/2}+e^{-sT/2}} = \frac{1}{s}\frac{e^{sT/2}-e^{-sT/2}}{e^{sT/2}(1+e^{-sT})}$

2 pts

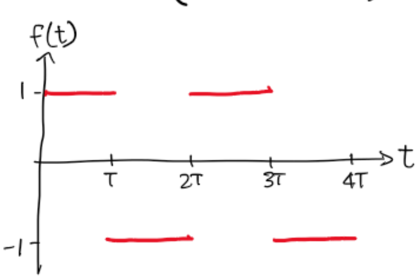
$$= \frac{1}{s}\frac{1-e^{-sT}}{1+e^{-sT}}$$

Let $F_0(s) = \frac{1-e^{-sT}}{s}$

$$f(t) = 1 \Rightarrow F_0(s) = \int_0^T e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^T = \frac{1}{s}\left(1-e^{-sT}\right)$$

so $f(t)=1$ satisfies this Laplace-transform for $t \in [0,T]$

Then $\mathbf{f(t)} = \begin{cases} 1 & t \in [k\pi,(k+1)\pi],\ k \text{ even} \\ -1 & t \in [k\pi,(k+1)\pi],\ k \text{ odd} \end{cases}$ (extend using periodic/odd conditions from (a))



$f(t)$ may also be written as $(-1)^{\lfloor t/T \rfloor}$ floor function

Q4 Let $U = \mathcal{F}\{u\}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\, dx$

4 pts

Then $u_{tt} + 2u_t + u = u_{xx}$ becomes:

$$U_{tt} + 2U_t + U = -k^2 U$$

$$\Rightarrow U_{tt} + 2U_t + (1+k^2)U = 0$$

Initial conditions:

$$u_t(x,0) = 0 \Rightarrow U_t(k,0) = 0$$

$$u(x,0) = \begin{cases} e^{-x} & x>0 \\ e^{x} & x<0 \end{cases} \Rightarrow U(k,0) = \int_0^{\infty} e^{-x-ikx}\, dx + \int_{-\infty}^{0} e^{x-ikx}\, dx$$

$$= \frac{1}{-1-ik} e^{-x-ikx}\Big|_0^{\infty} + \frac{1}{1-ik} e^{x-ikx}\Big|_{-\infty}^{0}$$

$$= \frac{1}{1+ik} + \frac{1}{1-ik} = \frac{2}{1+k^2}$$

Solve ODE in U:

$$U = e^{\omega t} \Rightarrow \omega^2 e^{\omega t} + 2\omega e^{\omega t} + (1+k^2)e^{\omega t} = 0$$

$$(\omega+1)^2 + k^2 = 0$$

$$\omega = \pm ki - 1$$

Then solutions of U are of the form

$$U(k,t) = a(k)e^{t(-1+ki)} + b(k)e^{t(-1-ki)}$$

$$U_t(k,t) = (-1+ki)a(k)e^{t(-1+ki)} + (-1-ki)b(k)e^{t(-1-ki)}$$

Then $(-1+ki)a(k) + (-1-ki)b(k) = 0$

$$a(k) + b(k) = \frac{2}{1+k^2}$$

$$ki(a(k) - b(k)) = \frac{2}{1+k^2}$$

$$a(k) - b(k) = \frac{2}{ki(1+k^2)}$$

$$a(k) = \frac{1}{1+k^2} + \frac{1}{ki(1+k^2)} = \frac{1+ki}{ki(1+k^2)} = \frac{1}{ki(1-ki)} = \frac{1}{ki} + \frac{1}{1-ki} = \frac{1}{(k+\frac{i}{2})^2+\frac{1}{4}}$$

$$b(k) = \frac{1}{1+k^2} - \frac{1}{ki(1+k^2)} = \frac{ki-1}{ki(1+k^2)} = \frac{-1}{ki(1+ki)} = -\frac{1}{ki} + \frac{1}{1+ki} = \frac{1}{(k-\frac{i}{2})^2+\frac{1}{4}}$$

$$U(k,t) = e^{-t}\left[\left(\frac{1}{ki} + \frac{1}{1-ki}\right)e^{tki} - \left(\frac{1}{ki} - \frac{1}{1+ki}\right)e^{-tki}\right]$$

Inverse Fourier Transform:

$$u(x,t) = e^{-t}\left[\mathcal{F}^{-1}\left\{\frac{1}{ki} + \frac{1}{1-ki}\right\}(x+t) - \mathcal{F}^{-1}\left\{\frac{1}{ki} - \frac{1}{1+ki}\right\}(x-t)\right]$$

Q5 (i) Apply the Nyquist criterion:

2 pts

$$N_0(p) = \frac{1}{2\pi}\left[3\pi + \Delta_\Gamma \arg\left(p(iy)\right)\right] \quad (\Gamma \text{ is the imaginary axis downwards})$$

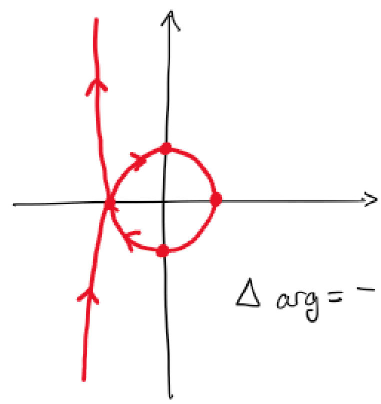
$$p(iy) = (iy)^3 + 2(iy)^2 + iy + 1$$
$$= \left(-2y^2+1\right) + i\left(-y^3+y\right)$$

Analyze each intersection point with real or imaginary axis:

Real intersections $= -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Imaginary intersections $= -1, 0, 1$

y	Re	Im
∞	$-$	$-$
1	$-$	0
$1/\sqrt{2}$	0	$+$
0	$+$	0
$-1/\sqrt{2}$	0	$-$
-1	$-$	0
$-\infty$	$-$	$+$



$\Delta \arg = -3\pi$

By Nyquist criterion, p has **no zeros** in the right half plane

(ii) $N_0(p) = \frac{1}{2\pi}\left[4\pi - \Delta_\Gamma \arg(p(iy))\right]$

2 pts

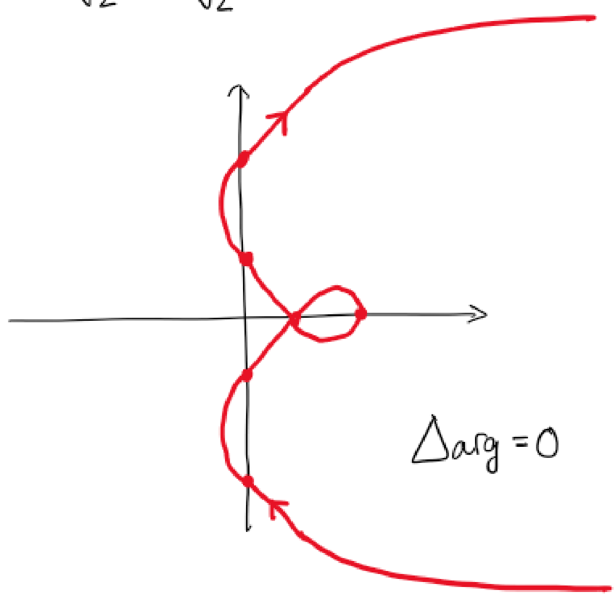
$$p(iy) = (iy)^4 + 2(iy)^3 + 3(iy)^2 + iy + 2$$

$$= \underbrace{(y^4 - 3y^2 + 2)}_{(y^2-2)(y^2-1)} + i(-2y^3 + y)$$

Real intersections $= -\sqrt{2}, -1, 1, \sqrt{2}$

Imaginary intersections $= -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

y	Re	Im
∞	+	−
$\sqrt{2}$	0	−
	−	−
1	0	−
$1/\sqrt{2}$	+	0
	+	+
0	+	0
	+	−
$-1/\sqrt{2}$	+	0
-1	0	+
	−	+
$-\sqrt{2}$	0	+
$-\infty$	+	+



$\Delta \arg = 0$

$\Rightarrow$ p has **2 zeros** in the right half plane